AUCTIONS WITH WEAKLY ASYMMETRIC INTERDEPENDENT VALUES

Gadi Fibich, Arieh Gavious and Aner Sela

Discussion Paper No. 03-09

July 2003

Monaster Center for Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Auctions with Weakly Asymmetric Interdependent Values

Gadi Fibich [*] Arieh Gavious [†] Aner Sela [‡]

June 23, 2003

Abstract

We study second price auctions with weakly asymmetric interdependent values where bidders' signals for the value are independently and identically distributed. We also prove an asymptotic revenue equivalence among all standard auctions with weakly asymmetric interdependent values.

Keywords: Weakly Asymmetric Auctions; Interdependent Values; Perturbation Analysis.

[*]School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel, fibich@math.tau.ac.il

[†]Department of Industrial Engineering and Management, Faculty of Engineering Sciences, Ben-Gurion University, P.O. Box 653, Beer-Sheva 84105, Israel, ariehg@bgumail.bgu.ac.il

[‡]Department of Economics, Ben-Gurion University, P.O. Box 653, Beer-Sheva 84105, Israel, anersela@bgumail.bgu.ac.il

1 Introduction

Bidders in auctions are symmetric if 1) their utility functions have the same form, and 2) they have the same beliefs about each other's values (i.e., their values or signals are identically distributed). Most of auction theory has been developed for situations in which bidders are symmetric, and does not extend to asymmetric bidders. In many cases in practice, however, a weak asymmetry can exist between bidders. For example, in the Personal Communication Spectrum (PCS) licenses sold by the US Government in the 1995 'Air-waves Auctions', each of the bidding telecommunications companies probably had similar, though not identical, values (Klemperer (1998)).

In this paper we consider auctions with asymmetric interdependent values, where bidders' signals are independently and identically distributed, and each bidder's signal is private information to that bidder. The value of each bidder depends on his signal and on the signals of the other bidders. In Section 2 we formulate a mathematical model in which the valuation of each bidder is written as $V_i = u + \epsilon U_i$, where u is a common valuation function, and ϵ is the level of asymmetry among bidders valuations. Since in the case of a weak asymmetry ϵ is small, perturbation methods can be employed to analyze the leading-order effect of asymmetry among bidders' valuations.

In Section 3 we use perturbation analysis to calculate an explicit approximation of the equilibrium strategies in second-price auctions with two asymmetric bidders. As in the symmetric setup, the approximate equilibrium strategies depend on the valuation functions, but not on the underlying distribution of signals. Unlike the symmetric case, however, the approximate equilibrium bid of every bidder in the asymmetric case is equal to his valuation assuming that his opponent has identical signal, only in the special case of almost private values. Thus, a weak asymmetry has the "smallest" effect when the symmetric value function u is a private value. In the other extreme case, namely, that of almost common value auctions (i.e., when u is the sum of all signals), a weak asymmetry has the "largest" effect, as the equilibrium strategies are considerably different from the symmetric ones. We end this section by using the approximate equilibrium strategies to calculate the seller's expected revenue in second-price auctions with weakly asymmetric interdependent values.

In Section 4, we consider the seller's revenue in any standard auction[1] with weakly asymmetric interdependent values. We recall that Myerson (1981) and Riley and Samuelson (1981) showed that all standard symmetric private-value auctions in which bidders values are independently distributed are revenue equivalent. Bulow and Klemperer (1996) generalized this result to the case of symmetric auctions with interdependent values, in which bidders signals are independently distributed.[2] It is well known, however, that standard auctions with asymmetric interdependent values are not necessarily revenue equivalent (see, e.g., Example 8.1 in Krishna, 2002). Nevertheless, we show that weakly asymmetric auctions with interdependent values in which bidders' signals are uncorrelated

[1]We say that an auction is standard if the rules of the auction dictate that the bidder with the highest bid wins the auction.

[2]Standard symmetric auctions with interdependent values are not necessarily revenue equivalent if the bidders' signals are affiliated. For example, the second-price auction generates more revenue than the first-price auction (Milgrom and Weber, 1982).

are asymptotically revenue equivalent, since weak asymmetry generates differences of only second order in revenues across auctions. Formally, let $R(\epsilon)$ be the seller's expected revenue in equilibrium. Then, $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where both $R(0)$, the seller's expected revenue in the symmetric setup where $V_i = u$, and $R'(0)$, the leading-order effect of the asymmetry, are independent of the auction mechanism.[3] From the expression for $R'(0)$ it follows that, independent of the underlying distribution of signals, the seller's revenue increases (decreases) with respect to the symmetric case ($\varepsilon = 0$) if the aggregation of the asymmetry of the bidders' valuations functions ($\sum_i U_i$) is positive (negative). Moreover, the seller's revenue in auctions with weakly asymmetric interdependent values can be approximated, with $O(\epsilon^2)$ accuracy, with the revenue in the case of symmetric auctions in which the valuation u is the arithmetic average of the original asymmetric valuations. Consequently, one can use the classical results of symmetric auctions in studying the seller's revenue in auctions with weakly asymmetric interdependent values.

2 The Model

Consider n risk-neutral bidders bidding for an indivisible object in a standard auction where the highest bidder wins the object. Bidder i, $i = 1, \ldots, n$ receives a signal x_i which is independently drawn from the interval $[0, 1]$ according to a common continuously-differentiable distribution function $F(x_i)$, with a corresponding density function $f = F'$.

[3] A similar result was obtained by Fibich et al. (2003) for private value auctions where the bidders have symmetric valuation functions but asymmetric beliefs about each other.

The signal x_i is private information to i. We denote by $\mathbf{x}_{-i}$ the $n-1$ signals other than x_i. The value of the object to bidder i, denoted by V_i, can be expressed as a function of all the bidders' signals, i.e.,

$$V_i(x_{i,}\mathbf{x}_{-i}) = u(x_{i,}\mathbf{x}_{-i}) + \epsilon U_i(x_{i,}\mathbf{x}_{-i}), \tag{1}$$

where V_i and u are assumed to be monotonically increasing in all their variables, are twice continuously differentiable, and satisfy $u(0,\ldots,0) = U_i(0,\ldots,0) = 0$. We also assume that u and U_i are *symmetric* in the $n-1$ components of $\mathbf{x}_{-i}$, i.e., from a bidder's point of view the signals of his opponents can be interchanged without affecting his value.

The assumption that the valuations are given by the form (1) is not restrictive. Indeed, consider the case of n bidders with valuation functions $\{V_i(x_i,\mathbf{x}_{-i})\}_{i=1}^n$, each of which is symmetric with respect to $\mathbf{x}_{-i}$. Let us first define the average (symmetric) valuation as

$$u(x_i,\mathbf{x}_{-i}) = \frac{1}{n}\sum_{k=1}^{n} V_k(x_i,\mathbf{x}_{-i}). \tag{2}$$

We also define

$$\epsilon = \max_i \max_{x_1,\ldots,x_n} |V_i - u|, \tag{3}$$

and

$$U_i(x_i,\mathbf{x}_{-i}) = \frac{V_i(x_i,\mathbf{x}_{-i}) - u(x_i,\mathbf{x}_{-i})}{\epsilon}. \tag{4}$$

Then, the V_i's are given by the form (1), with u, ϵ, and U_i given by (2,3,4).

The parameter ϵ is the measure of the asymmetry among players' valuations. In particular, $\epsilon \ll 1$ corresponds to the case of *auctions with weakly asymmetric interdependent values.*

3 Second-Price Auctions

Consider a second-price auction, in which the bidder with the highest bid wins the object and pays the second highest bid. For simplicity we consider the case of two bidders, i.e.,

$$V_1(x_1, x_2) = u(x_1, x_2) + \epsilon U_1(x_1, x_2), \qquad V_2(x_2, x_1) = u(x_2, x_1) + \epsilon U_2(x_2, x_1). \tag{5}$$

The expected utility of bidder i, $i = 1, 2$, with signal x who makes a bid b is given by

$$EU_1(b, x) = \int_0^{b_2^{-1}(b)} (V_1(x, s) - b_2(s))\, f(s)\, ds, \quad EU_2(b, x) = \int_0^{b_1^{-1}(b)} (V_2(x, s) - b_1(s))\, f(s)\, ds.$$

The inverse equilibrium strategies $x_i(b) = b_i^{-1}(b)$, $i = 1, 2$ are determined from

$$\frac{\partial EU_1(b, x)}{\partial b} = \frac{\partial EU_2(b, x)}{\partial b} = 0,$$

leading to the system

$$V_1(x_1(b), x_2(b)) = b, \qquad V_2(x_2(b), x_1(b)) = b. \tag{6}$$

In the following we employ perturbation analysis to calculate the leading-order effect of asymmetry on the equilibrium strategies:

Proposition 1 *Assume that for all x, the symmetric valuation function u satisfies $u_{x_1}(x, x) \neq u_{x_2}(x, x)$. Then, the equilibrium strategies in a two-player second-price auction with weakly asymmetric interdependent values (5) are given by*

$$\begin{aligned} b_1(x) &= u(x, x) + \epsilon \frac{U_1 u_{x_1} - U_2 u_{x_2}}{u_{x_1} - u_{x_2}}(x, x) + O(\epsilon^2), \\ b_2(x) &= u(x, x) + \epsilon \frac{U_2 u_{x_1} - U_1 u_{x_2}}{u_{x_1} - u_{x_2}}(x, x) + O(\epsilon^2). \end{aligned} \tag{7}$$

Proof: See Appendix A. ⊡

Example 1 *Consider a second-price auction with weakly asymmetric interdependent values, where the signals of the two bidders are uniformly distributed in* $[0,1]$*, and the bidders valuations functions are given by*

$$V_1 = x_1, \qquad V_2 = x_2 + \epsilon x_1 x_2.$$

Hence,

$$u(x_1, x_2) = x_1, \qquad U_1 = 0, \qquad U_2 = x_1 x_2.$$

The equilibrium bid function of each bidder in the symmetric case ($\epsilon = 0$) *is given by* $b(x) = u(x,x) = x$*. From (7), the bids in the case of weak asymmetry are given by*

$$b_1 = x + O(\epsilon^2), \qquad b_2 = x + \epsilon x^2 + O(\epsilon^2).$$

In this case we can also calculate the bids exactly. From (6), the (exact) equilibrium equations are

$$x_1 = b, \qquad x_2 + \epsilon x_1 x_2 = b,$$

which gives the inverse equilibrium bids

$$x_1 = b \quad and \quad x_2 = \frac{b}{1+\epsilon b},$$

and accordingly the (exact) bidding functions are

$$b_1 = x, \qquad b_2 = \frac{x}{1-\epsilon x}.$$

We thus see that the approximation for b_1 is exact. The approximation for b_2 agrees with the exact expression to second-order in ϵ, since $x/(1-\epsilon x) = x + \epsilon x^2 + O(\epsilon^2)$. In fact, the two seem to be in good agreement even at a 25% asymmetry level (see Figure 1).

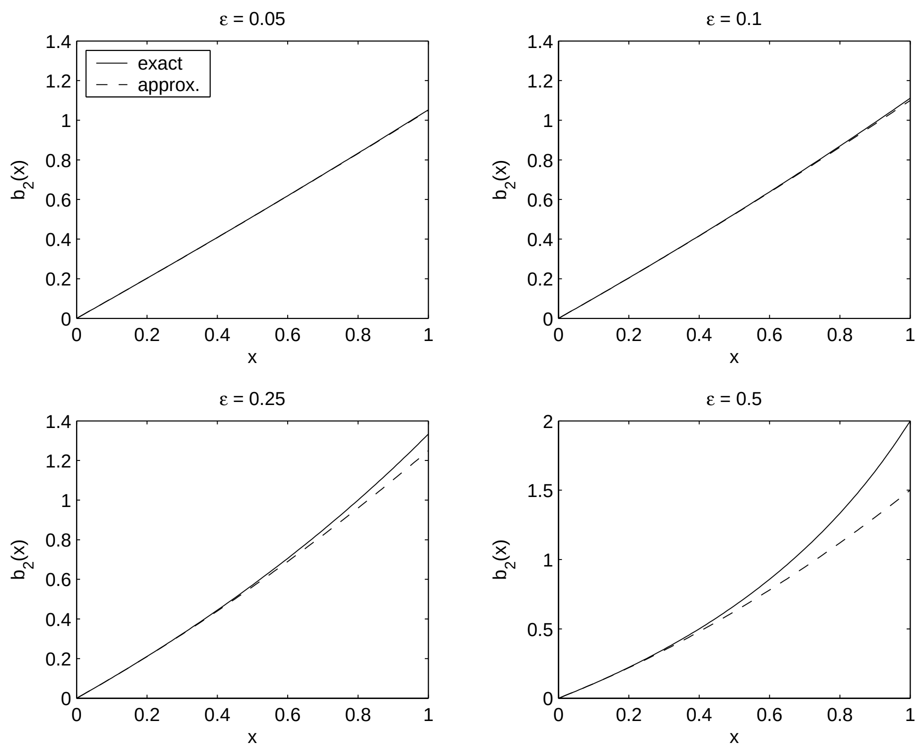

Figure 1: Exact and approximate equilibrium bids of second player in Example 1.

We recall that in the symmetric setup, Milgrom (1981) and Milgrom and Weber (1982) showed that the equilibrium bid of each bidder is equal to his valuation given that his opponent has an identical signal, i.e., $b_i(x) = u(x, x)$, $i = 1, 2$. We now show that under weak asymmetry, this property remains true[4] only in the special case of "almost private values" where $u(x_1, x_2) = u(x_1)$:

[4] To leading order in ϵ.

Lemma 1 *The equilibrium strategies in a second-price auction with weakly asymmetric interdependent values (5) are given by* $b_i(x) = V_i(x,x) + O(\epsilon^2)$, $i = 1, 2$, *if and only if* $u_{x_2} \equiv 0$.

Proof. This follows by rewriting (7) as

$$\begin{aligned} b_1(x) &= V_1(x,x) + \epsilon u_{x_2} \frac{U_1 - U_2}{u_{x_1} - u_{x_2}}(x,x) + O(\epsilon^2), \\ b_2(x) &= V_2(x,x) + \epsilon u_{x_2} \frac{U_2 - U_1}{u_{x_1} - u_{x_2}}(x,x) + O(\epsilon^2). \end{aligned}$$

⊡

In the case of "almost common values," i.e., $u(x_1, x_2) = x_1 + x_2$, the perturbation analysis in Proposition 1 "fails" to calculate the equilibrium bids. It can be shown that this "failure" is not related to the fact that the solution in Proposition 1 is a small perturbation of the symmetric equilibrium bids $u(x,x)$, and not of one of the asymmetric equilibria in symmetric second-price auctions (see, Milgrom (1981), Bikhchandani (1988), and Bikhchandani and Riley (1991)). Rather, this "failure" shows that either the equilibrium bids in "almost common value auctions" do not exist, or that if they do exist, they are not small perturbations of the equilibria in the symmetric setup. Indeed, let us consider the asymmetric Wallet-Game model (see Klemperer (1998)) where

$$V_1 = x_1 + x_2 + \epsilon, \qquad V_2 = x_2 + x_1.$$

From (6), the equilibrium system is

$$V_1(x_1, x_2) = x_1 + x_2 + \epsilon = b, \qquad V_2(x_2, x_1) = x_2 + x_1 = b.$$

Obviously, there is no solution to this system of equations.[5]

Using the equilibrium strategies (7), we can calculate an approximation of the seller's expected revenue which we denote by $R(\epsilon)$:

Proposition 2 *Consider a second price auction with two bidders with weakly asymmetric interdependent values given by (5). Then, the seller's expected revenue is given by* $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, *where*

$$R(0) = 2\int_0^1 u(x,x)(1-F(x))f(x)\,dx, \tag{8}$$

and

$$R'(0) = \int_0^1 \left[U_1(x,x) + U_2(x,x)\right](1-F(x))f(x)\,dx\ . \tag{9}$$

Proof: See Appendix B.

Example 2 *We calculate the exact and approximate expressions for the seller's expected revenue from Example 1. By (17,18), the exact expected revenue in the second price auction is given by*

$$R = \bar{b} - \int_0^{\bar{b}} \left[F(x_1(b)) + F(x_2(b)) - F(x_1(b))F(x_2(b))\right]\,db,$$

where $\bar{b}$ *is the maximal price, or the second-highest bid, in equilibrium. We have already seen that* $x_1 = b$ *and* $x_2 = \frac{b}{1+\epsilon b}$. *Hence,* $\bar{b} = 1$, *and the exact expected revenue is given by*

$$R = 1 - \int_0^1 \left(b + \frac{b}{1+\epsilon b} - \frac{b^2}{1+\epsilon b}\right)db = \frac{1}{2} - \frac{1}{2\epsilon} - \frac{1}{\epsilon^2} + \frac{\ln(1+\epsilon)}{\epsilon^2} + \frac{\ln(1+\epsilon)}{\epsilon^3}. \tag{10}$$

[5] An equilibrium does exist in the case of an English auction (Klemperer, 1998).

If we calculate the approximation for expected revenue obtained from (8) and (9), we obtain

$$R = 1/3 + (1/12)\epsilon + O(\epsilon^2).$$

It is easy to verify that Taylor expansion of (10) also gives $R = 1/3 + (1/12)\epsilon + O(\epsilon^2)$*, i.e, the two agree to* $O(\epsilon^2)$*. Indeed, Figure 2 shows that the perturbation analysis approximation of the seller's expected revenue is in good agreement with the exact value for* ϵ *up to 0.3.*

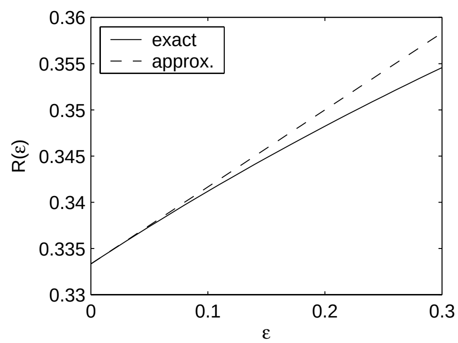

Figure 2: Exact and approximate values of the seller's expected revenue in Example 2.

In the next section we show that the expression that we derived for the seller's expected revenue in second-price auctions with two bidders can be generalized to all standard auctions with any number of bidders.

4 Revenue Equivalence

According to the Revenue Equivalence Theorem (see Vickrey (1961), Myerson (1981) and Riley and Samuelson (1981)) the expected revenue of the seller in equilibrium is independent of the auction mechanism whenever bidders' values are independent of each other, bidders are risk neutral, and particularly, bidders are ex-ante symmetric. Bulow and Klemperer (1996) generalized the Revenue Equivalence Theorem to symmetric auctions with interdependent values where the bidders' signals are independent. Myerson (1981) showed that the revenue equivalence also holds for asymmetric auctions, provided that at any realization of the players' valuations the probability of a player to win the object is independent of the auction mechanism. However, even in weakly asymmetric auctions it can be easily verified that Myerson's condition usually does not hold and therefore asymmetric auctions are not necessarily revenue equivalent.

We now prove an asymptotic revenue equivalence among all asymmetric auctions with interdependent values, under the same conditions used in the (symmetric) Revenue Equivalence Theorem (Myerson, 1981; Riley and Samuelson, 1981; Bulow and Klemperer, 1996). Therefore, we consider auction mechanisms with n players and weakly asymmetric interdependent values that satisfy the following conditions:

Condition 1 All players are risk neutral.

Condition 2 The signal of player i is private information to i and is drawn independently by a continuously differentiable distribution function $F(x)$ from a support $[0, 1]$ which is common to all players.

Condition 3 The object is allocated to the player with the highest bid.[6]

[6]In the symmetric setup, Condition 3 is equivalent to the condition that the object is allocated to the player with the highest valuation. This equivalence, however, does not hold in the asymmetric setup

Condition 4 Any player i with signal $x_i = 0$ expects zero surplus.

Under these conditions, the following asymptotic revenue equivalence holds:

Theorem 1 *Consider any auction mechanism that satisfies Conditions 1–4, with n bidders having weakly-asymmetric interdependent values (1). Then, the seller's expected revenue is given by*

$$R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2),$$

where

$$R(0) = n(n-1)\int_0^1 E_{\mathbf{x_{-1,-2}}}\left[u(x_1 = x, x_2 = x, \mathbf{x_{-1,-2}})\middle| 1 \text{ wins}\right](1-F(x))f(x)\,dx, \quad (11)$$

$$R'(0) = (n-1)\int_0^1 \sum_{i=1}^{n} E_{\mathbf{x_{-i,-k}}}\left[U_i(x_i = x, x_k = x, \mathbf{x_{-i,-k}})\middle| i \text{ wins}\right](1-F(x))f(x)\,dx, \quad (12)$$

$k = k(i)$ is an arbitrary[7] index different than i, and $\mathbf{x_{-i,-j}}$ is the vector of signals not including x_i and x_j.

Proof: See Appendix C. ⊡

The revenue equivalence theorem for symmetric auctions with interdependent values (Bulow and Klemperer, 1996) implies that $R(0)$ is independent of the auction mechanism. The novelty in Theorem 1 is, thus, in showing that $\epsilon R'(0)$, the leading-order effect of asymmetry, is also independent of the auction mechanism. Hence, for a weak asymmetry the revenue difference among auctions with interdependent values is only second-order

since asymmetric auctions are not necessarily efficient.

[7] e.g., $k(i) = n$ if $i < n$ and $k(i) = 1$ if $i = n$.

in ϵ. In many cases these differences are only in the third or fourth digit, in which case the ranking among these auctions is more a matter of academic interest than of practical value.

In the case of two bidders, the expressions in Theorem 1 simplify as follows:

Conclusion 1 *Consider any two-bidder auction that satisfies Conditions 1–4, with weakly asymmetric interdependent values (5). Then, the seller's expected revenue is given by* $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$*, where* $R(0)$ *and* $R'(0)$ *are given by (8,9).*

Theorem 1 leads to the following comparison of the seller's expected revenue in asymmetric and symmetric auctions:

Conclusion 2 *Let* $\sum_{i=1}^{n} U_i(x_i = x, x_k = x, \mathbf{x_{-i,-k}}) \geq 0$ (≤ 0)*, where* k *is an arbitrary index different from* i*. Then the seller's revenue in the asymmetric case is larger (smaller) than his revenue in the symmetric case, i.e.,* $R(\varepsilon) \geq R(0)$ $(\leq R(0))$.

Remark. In the case of two bidders, the condition $\sum_{i=1}^{n} U_i(x_i = x, x_k = x, \mathbf{x_{-i,-k}}) \geq 0$ reduces to $U_1(x,x) + U_2(x,x) \geq 0$.

An immediate, yet important, consequence of Theorem 1 is that the seller's expected revenue in asymmetric auctions with n bidders can be well-approximated with the seller's expected revenue in the symmetric case with n bidders whose valuation function is the arithmetic average of the n asymmetric valuation functions:

Conclusion 3 *Consider any auction mechanism that satisfies Conditions 1–4, with* n *bidders having weakly-asymmetric interdependent values* $\{V_i\}_{i=1}^{n}$*. Denote the seller's expected* *revenue by* $R[V_1,\dots,V_n]$*. Then,*

$$R[V_1,\dots,V_2] = R_{\rm sym}[u] + O(\epsilon^2),$$

where $R_{\rm sym}[u]$*, the seller's expected revenue in the symmetric auction with* n *players with valuation function* u*, is given by (11), and* u *and* ϵ *are given by (2,3).*

Proof. When u is given by (2), it follows from (12) that $R'(0) = 0$. ⊡

Example 3 *We compare between the expected revenue of a first-price auction, the expected revenue of a second price auction, and the symmetric approximation* $R_{sym}[u]$*, using the valuation functions from Example 1.*

The expected revenue in a second-price auction is given by (10). The expected revenue in a first-price auction is calculated numerically (for details, see Appendix D). By (2), the average valuation is

$$u(x_1, x_2) = x_1 + 0.5\epsilon x_1 x_2.$$

Therefore, substitution in (8) gives the symmetric approximation of the revenue

$$R_{\rm sym}[u] \approx 1/3 + \epsilon/12.$$

As Table 1 shows, the differences among the expected revenue of a first-price auction, the expected revenue of a second price auction, and the symmetric approximation $R_{sym}[u]$ *are only in the third or fourth digit. Moreover, it is easy to see that these differences scale like* ϵ^2*, since doubling the value of* ϵ *leads to a four-fold increase in the revenue difference.*

ϵ	R^{first}	R^{second}	$R[u_{sym}]$	$\frac{R^{\text{first}}-R^{\text{second}}}{R^{\text{first}}}100\%$	$\frac{R^{first}-R[u_{sym}]}{R^{first}}100\%$
0.05	0.33749	0.33738	0.33750	0.03%	0.003%
0.1	0.34161	0.34120	0.34166	0.12%	0.015%
0.2	0.34979	0.34823	0.35000	0.46%	0.06%

Table 1: Seller's expected revenue in first- and second-price auctions.

References

[1] Bikhchandani, S. (1988): "Reputation in Repeated Second-Price Auctions," *Journal of Economic Theory,* 46, 97-119.

[2] Bikhchandani, S. and Riley, J. (1991): "Equilibria in Open Common Value Auctions," *Journal of Economic Theory,* 53, 101-130.

[3] Bulow, J. I. and Klemperer, P. D. (1996): "Auctions Versus Negotiations," *American Economic Review,* 86, 1, 180-194.

[4] Fibich, G., Gavious, A. and Sela, A. (2003): "Revenue Equivalence in Asymmetric Auctions," forthcoming in *Journal of Economic Theory.*

[5] Fibich, G. and Gavious, A. (2003): "Asymmetric First-Price Auctions - A Perturbation Approach," forthcoming in *Mathematics of Operations Research.*

[6] Klemperer, P. D. (1998): "Auctions with Almost Common Values," *European Economic Review*, 42, 757-769.

[7] Krishna, V. (2002), "Auction Theory", Academic Press, London.

[8] Marshall, R. C., Meurer, M. J., Richard, J.-F. and Stromquist, W. (1994), "Numerical Analysis of asymmetric First Price Auctions", *Games and Economic behavior,* 7, 193-220.

[9] Maskin, E. S. and Riley, J. G. (2000): "Asymmetric Auctions," *Review of Economic Studies,* 67, 413-438.

[10] Milgrom, P. (1981): "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica,* 49, 921-943.

[11] Milgrom, P. and Weber, R. (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica,* 50, 1089-1122.

[12] Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research,* 6, 58-73.

[13] Riley, J. G. and Samuelson, W. F. (1981): "Optimal Auctions," *American Economic Review,* 71, 381-392.

[14] Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance,* 16, 8-37.

A Proof of Proposition 1

We assume that when ε is sufficiently small (i.e., weak asymmetry) the auction has equilibrium bids $\{b_i(x;\epsilon)\}_{i=1}^n$ that are strictly increasing functions of x and continuously differentiable with respect to ε. Let us expand the equilibrium bids in ϵ, i.e.,

$$b_i(x) = b_{sym}(x) + \epsilon B_i(x) + O(\epsilon^2), \quad i = 1, 2 \ , \tag{13}$$

where $b_{sym}(x)$ is the equilibrium bid in the symmetric case. Similarly, we expand the inverse equilibrium bids in ϵ, i.e.,

$$x_i(b) = x_{sym}(b) + \epsilon X_i(b) + O(\epsilon^2), \quad i = 1, 2. \tag{14}$$

Substituting (14) in (1) gives

$$\begin{aligned} V_1(x_1(b), x_2(b)) &= u(x_1(b), x_2(b)) + \epsilon U_1(x_1(b), x_2(b)) = \qquad (15) \\ &= u(x_{sym}(b) + \epsilon X_1, x_{sym}(b) + \epsilon X_2) + \epsilon U_1(x_{sym}(b) + \epsilon X_1, x_{sym}(b) + \epsilon X_2) + O(\epsilon^2) \\ &= u(x_{sym}(b), x_{sym}(b)) + \epsilon X_1 u_{x_1}(x_{sym}(b), x_{sym}(b)) + \epsilon X_2 u_{x_2}(x_{sym}(b), x_{sym}(b)) \\ &\quad + \epsilon U_1(x_{sym}(b), x_{sym}(b)) + O(\epsilon^2). \end{aligned}$$

Similarly,

$$\begin{aligned} V_2(x_2(b), x_1(b)) &= u(x_{sym}(b), x_{sym}(b)) + \epsilon X_2 u_{x_1}(x_{sym}(b), x_{sym}(b)) + \epsilon X_1 u_{x_2}(x_{sym}(b), x_{sym}(b)) \\ &\quad + \epsilon U_2(x_{sym}(b), x_{sym}(b)) + O(\epsilon^2). \qquad (16) \end{aligned}$$

Substituting (15,16) in (6) gives

$$\begin{aligned} & u(x_{sym}(b), x_{sym}(b)) + \epsilon X_1 u_{x_1}(x_{sym}(b), x_{sym}(b)) + \epsilon X_2 u_{x_2}(x_{sym}(b), x_{sym}(b)) + \epsilon U_1(x_{sym}(b), x_{sym}(b)) \\ = \ & b + O(\epsilon^2), \\ & u(x_{sym}(b), x_{sym}(b)) + \epsilon X_2 u_{x_1}(x_{sym}(b), x_{sym}(b)) + \epsilon X_1 u_{x_2}(x_{sym}(b), x_{sym}(b)) + \epsilon U_2(x_{sym}(b), x_{sym}(b)) \\ = \ & b + O(\epsilon^2). \end{aligned}$$

We now expand both sides of the above equations in powers of ϵ, and equate the corresponding coefficients. The equation for the $O(1)$ terms is $u(x_{sym}(b), x_{sym}(b)) = b$, and its solution is given by $b_{sym}(x) = u(x, x)$. The equations for the $O(\epsilon)$ terms are given by

$$\begin{aligned} X_1 u_{x_1}(x_{sym}(b), x_{sym}(b)) + X_2 u_{x_2}(x_{sym}(b), x_{sym}(b)) &= -U_1(x_{sym}(b), x_{sym}(b)), \\ X_2 u_{x_1}(x_{sym}(b), x_{sym}(b)) + X_1 u_{x_2}(x_{sym}(b), x_{sym}(b)) &= -U_2(x_{sym}(b), x_{sym}(b)). \end{aligned}$$

The solution of this system is

$$X_1 = \frac{-U_1 u_{x_1} + U_2 u_{x_2}}{u_{x_1}^2 - u_{x_2}^2}, \qquad X_2 = \frac{-U_2 u_{x_1} + U_1 u_{x_2}}{u_{x_1}^2 - u_{x_2}^2}.$$

Since $B_i(x) = -b'_{sym}(x) X_i(b_{sym}(x)) = -\left(u_{x_1}(x,x) + u_{x_2}(x,x)\right) X_i(b_{sym}(x))$, we have

$$B_1 = \frac{U_1 u_{x_1} - U_2 u_{x_2}}{u_{x_1} - u_{x_2}}(x, x), \qquad B_2 = \frac{U_2 u_{x_1} - U_1 u_{x_2}}{u_{x_1} - u_{x_2}}(x, x).$$

B Proof of Proposition 2

The distribution of the second highest bid b is given by

$$\begin{aligned} F^{2nd}(b) &= \Pr(\min(b_1, b_2) \le b) = \Pr(\{b_1 \le b\} \cup \{b_2 \le b\}) \qquad (17) \\ &= \Pr(b_1 \le b) + \Pr(b_2 \le b) - \Pr(b_1 \le b, b_2 \le b) \\ &= \Pr(x_1 \le b_1^{-1}(b)) + \Pr(x_2 \le b_2^{-1}(b)) - \Pr(x_1 \le b_1^{-1}(b), x_2 \le b_2^{-1}(b)) \\ &= F(x_1(b)) + F(x_2(b)) - F(x_1(b) F(x_2(b)). \end{aligned}$$

Therefore, the seller's expected revenue is given by

$$R = \int_0^{\bar{b}} b \, dF^{2nd}(b) = bF^{2nd}(b)\Big|_0^{\bar{b}} - \int_0^{\bar{b}} F^{2nd}(b)\, db = \bar{b} - \int_0^{\bar{b}} F^{2nd}(b)\, db, \qquad (18)$$

where $\bar{b}$ is the maximal price, or the second-highest bid, in equilibrium. Since by (14),

$$F(x_i(b)) = F(x_{sym}(b) + \epsilon X_i + O(\epsilon^2)) = F(x_{sym}(b)) + \epsilon X_i f(x_{sym}(b)) + O(\epsilon^2),$$

it follows that

$$F^{2nd} = F^{2nd}_{sym} + \epsilon[X_1 + X_2] f(x_{sym}) \, [1 - F(x_{sym})] + O(\epsilon^2).$$

Let us expand the maximal price in ϵ, i.e., $\bar{b}(\epsilon) = \bar{b}_{sym} + \epsilon \bar{b}_1 + \epsilon^2$. Since $F^{2nd}(\bar{b}) = 1$, then $F^{2nd}(\bar{b}_{sym}) = 1 + O(\epsilon)$. Therefore,

$$\begin{aligned} R &= \bar{b} - \int_0^{\bar{b}} F^{2nd}(b)\, db = \bar{b}_{sym} + \epsilon \bar{b}_1 - \int_0^{\bar{b}_{sym}} F^{2nd}(b)\, db - \epsilon \bar{b}_1 F^{2nd}(\bar{b}_{sym}) + O(\epsilon^2) \\ &= \bar{b}_{sym} - \int_0^{\bar{b}_{sym}} F^{2nd}(b)\, db + O(\epsilon^2). \end{aligned}$$

Combining the above, we get that

$$\begin{aligned} R &= \bar{b}_{sym} - \int_0^{\bar{b}_{sym}} F^{2nd}_{sym}(b)\, db - \epsilon \int_0^{\bar{b}_{sym}} [X_1 + X_2] \, f(x_{sym}(b)) \, [1 - F(x_{sym}(b))] \, db + O(\epsilon^2) \\ &= R_{sym} - \epsilon \int_0^{\bar{b}_{sym}} [X_1 + X_2] \, f(x_{sym}(b)) \, [1 - F(x_{sym}(b))] \, db + O(\epsilon^2) \\ &= R_{sym} + \epsilon \int_0^{\bar{b}_{sym}} \frac{U_1 + U_2}{u_{x_1} + u_{x_2}} f(x_{sym}(b)) \, [1 - F(x_{sym}(b))] \, db + O(\epsilon^2). \end{aligned}$$

Since $b_{sym}(x) = u(x,x)$ we have $x'_{sym}(b) = \frac{1}{u_{x_1} + u_{x_2}}$ and thus, by changing variable we have

$$R = R_{sym} + \epsilon \int_0^1 (U_1(x,x) + U_2(x,x)) \, f(x) \, [1 - F(x)] \; dx + O(\epsilon^2).$$

C Proof of Theorem 1

Let $E_i(x)$, $S_i(x)$, and $P_i(x)$ be the expected payment, the expected surplus, and the probability of winning for bidder i with signal x at equilibrium, respectively. Then,[8]

$$S_1(x_1) = E_{\mathbf{x_{-1}}}\left[V_1(x_1, \mathbf{x_{-1}}) \mid 1 \text{ wins}\right] - E_1(x_1), \tag{19}$$

where $\mathbf{x_{-1}} = (x_2, \ldots, x_n)$ and

$$E_{\mathbf{x_{-1}}}\left[V_1(x_1, \mathbf{x_{-1}}) \middle| 1 \text{ wins}\right] = \int_{x_2=0}^{b_2^{-1}(b_1(x_1))} \cdots \int_{x_n=0}^{b_n^{-1}(b_1(x_1))} V_1(x_1, \mathbf{x_{-1}}) f(x_2) \cdots f(x_n)\, dx_2 \cdots dx_n. \tag{20}$$

Applying a standard argument (see, e.g., Bulow and Klemperer (1996) and Klemperer (1999)), for any $\tilde{x_1} \neq x_1$,

$$S_1(x_1) \geq S_1(\tilde{x_1}) - E_{\mathbf{x_{-1}}}[V_1(\tilde{x_1}, \mathbf{x_{-1}}) - V_1(x_1, \mathbf{x_{-1}}) \mid 1 \text{ wins}].$$

Therefore,

$$S_1(\tilde{x_1}) - S_1(x_1) \leq E_{\mathbf{x_{-1}}}[V_1(\tilde{x_1}, \mathbf{x_{-1}}) - V_1(x_1, \mathbf{x_{-1}}) \mid 1 \text{ wins}].$$

Substituting $\tilde{x_1} = x_1 + dx$ with $dx > 0$, dividing both sides by dx and letting $dx \longrightarrow 0$ gives

$$S_1'(x_1) \leq E_{\mathbf{x_{-1}}}\left[\frac{\partial V_1}{\partial x_1} \mid 1 \text{ wins}\right].$$

Repeating this procedure with $dx < 0$ gives

$$S_1'(x_1) \geq E_{\mathbf{x_{-1}}}\left[\frac{\partial V_1}{\partial x_1} \mid 1 \text{ wins}\right].$$

[8] To simplify the notations, we work with S_1 rather than with S_i.

Hence,

$$S_1'(x_1) = E_{\mathbf{x_{-1}}}\left[\frac{\partial V_1}{\partial x_1} \mid 1 \text{ wins}\right]. \tag{21}$$

Differentiating (19) with respect to x_1, substituting (21) and using (20) gives

$$\begin{aligned}
E_1'(x_1) &= \frac{d}{dx_1} E_{\mathbf{x_{-1}}}\left[V_1(x_1, \mathbf{x_{-1}}) \mid 1\text{wins}\right] - S_1'(x_1) \\
&= \sum_{j=2}^{n} \frac{d}{dx_1}\left[b_j^{-1}(b_1(x_1))\right] E_{\mathbf{x_{-1,-j}}}\left[V_1(x_1, b_j^{-1}(b_1(x_1)), \mathbf{x_{-1,-j}}) \mid 1 \text{ wins}\right] f(b_j^{-1}(b_1(x_1))),
\end{aligned}$$

where $\mathbf{x_{-1,-j}}$ is $\mathbf{x_{-1}}$ without the x_j element. Similarly, for player i,

$$E_i'(x_i) = \sum_{j=1, j\neq i}^{n} \frac{d}{dx_i}\left[b_j^{-1}(b_i(x_i))\right] E_{\mathbf{x_{-i,-j}}}\left[V_i(x_i, b_j^{-1}(b_i(x_i)), \mathbf{x_{-i,-j}}) \mid i \text{ wins}\right] f(b_j^{-i}(b_i(x_i))), \tag{22}$$

where $\mathbf{x_{-i,-j}}$ is $(x_1, x_2, \ldots, x_n)$ without the x_i and x_j elements.

Let $R_i(\epsilon)$ be the expected payments of player i averaged across her signals. Then,

$$\begin{aligned}
R_i(\epsilon) &= \int_0^1 E_i(x) f(x)\, dx = E_1(x) F\Big|_0^1 - \int_0^1 E_i'(x) F(x)\, dx \qquad (23) \\
&= E_i(1) - \int_0^1 E_i'(x) F(x)\, dx = E_i(0) + \int_0^1 E_i'(x)(1 - F(x))\, dx.
\end{aligned}$$

Let us denote $B_{j,i}(x) = b_j^{-1}(b_i(x))$. Therefore, from (22,23) we have that

$$\begin{aligned}
R_i(\epsilon) &= E_i(0;\epsilon) + \\
&\int_0^1 \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{d}{dx}\left[B_{j,i}(x)\right] E_{\mathbf{x_{-i,-j}}}\left[V_i(x_i = x, x_j = B_{j,i}(x), \mathbf{x_{-i,-j}}) \mid i \text{ wins}\right] f(B_{j,i}(x))(1 - F(x))\, dx.
\end{aligned}$$

The seller's expected revenue is given by $R(\epsilon) = \sum_{i=1}^{n} R_i(\epsilon)$. In the symmetric case $(\epsilon = 0)$, $B_{j,i}(x) = x$. In addition, from (19) and Condition 4 it follows that $E_i(x = 0; \epsilon =$

0) = 0. Therefore, the expected revenue in the symmetric case is given by $R(0) = nR_1(0)$, where

$$
\begin{aligned}
R_1(0) &= \sum_{j=2}^{n} \int_0^1 E_{\mathbf{x_{-1,-j}}} \left[u(x_1, x_j = x_1, \mathbf{x_{-1,-j}}) \;\middle|\; 1 \text{ wins} \right] f(x_1)(1 - F(x_1))\, dx_1 \\
&= (n-1) \int_0^1 E_{\mathbf{x_{-1,-2}}} \left[u(x_1, x_2 = x_1, \mathbf{x_{-1,-2}}) \;\middle|\; 1 \text{ wins} \right] f(x_1)(1 - F(x_1))\, dx_1,
\end{aligned}
$$

which leads to (11). Note that the last equality follows from the symmetry of u (Section 2).

We now proceed to calculate $R'(0) = \sum_{i=1}^{n} R_i'(0)$. We have that $R_i'(0) = I_{i,1} + I_{i,2} + I_{i,3}$, where

$$
\begin{aligned}
I_{i,1} &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_i(0;\epsilon), \\
I_{i,2} &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left\{ \int_0^1 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{d}{dx}\Big[B_{j,i}(x) \Big] E_{\mathbf{x_{-i,-j}}} \left[u(x_i, B_{j,i}(x), \mathbf{x_{-i,-j}}) | i \text{ wins} \right] f(B_{j,i}(x))(1 - F(x))\, dx \right\}, \\
I_{i,3} &= \int_0^1 \sum_{\substack{j=1 \\ j \neq i}}^{n} E_{\mathbf{x_{-i,-j}}} \left[U_i(x_i = x, x_j = x, \mathbf{x_{-i,-j}}) | i \text{ wins} \right] f(x)(1 - F(x))\, dx.
\end{aligned}
$$

The proof follows from the fact that

$$
I_{i,1} = 0, \quad i = 1, \ldots, n. \tag{24}
$$

and

$$
\sum_{i=1}^{n} I_{i,2} = 0. \tag{25}
$$

Therefore, $R'(0) = \sum_{i=1}^{n} I_{i,3}$. Utilizing the symmetry of U_i (see Section 2) gives (11).

To prove (24), we different (19) with respect to ϵ, substitute $\epsilon = 0$ and use Condition 4 to get that

$$
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_1(0;\epsilon) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_{\mathbf{x_{-1}}} \left[V_1(x_1 = 0, \mathbf{x_{-1}}) \;\middle|\; 1 \text{ wins} \right].
$$

From (20) it follows that when there are more than two players,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_{\mathbf{x_{-1}}}\left[V_1(x_1=0,\mathbf{x_{-1}}) \;\middle|\; 1 \text{ wins}\right] = 0,$$

since at least one of the integrals goes from zero to zero. When there are only two players,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_{\mathbf{x_{-1}}}\left[V_1(x_1=0,x_2) \;\middle|\; 1 \text{ wins}\right] = \left.\frac{\partial B_{2,1}(0)}{\partial\epsilon}\right|_{\epsilon=0} u(0,0)f(0) = 0,$$

since $u(0,0)=0$. Therefore,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_1(0;\epsilon) = 0.$$

Since the above can be repeated to all other players, we have proved (24).

To prove (25), first note that

$$\left.\frac{\partial B_{j,i}}{\partial\epsilon}\right|_{\epsilon=0} = (b_{sym}^{-1})'\frac{\partial b_i(x)}{\partial\epsilon} + \left.\frac{\partial b_j^{-1}(x)}{\partial\epsilon}\right|_{\epsilon=0}.$$

Differentiating the identity $x = b_j^{-1}(b_j(x;\epsilon);\epsilon)$ with respect to ϵ and substituting $\epsilon=0$ gives

$$0 = \left.\frac{\partial b_j^{-1}}{\partial\epsilon}\right|_{\epsilon=0} + (b_{sym}^{-1})'\left.\frac{\partial b_j}{\partial\epsilon}\right|_{\epsilon=0},$$

and hence

$$\frac{\partial}{\partial\epsilon}\left[B_{i,j}+B_{j,i}\right]_{\epsilon=0} = 0. \tag{26}$$

Since

$$I_{i,2} = \int_0^1 G(x)\sum_{\substack{j=1\\ j\neq i}}^{n} \left.\frac{dB_{j,i}(x)}{d\epsilon}\right|_{\epsilon=0} dx,$$

where $G(x)$ is independent of the index i, application of (26) proves (25).

D Expected revenue in first-price auctions (Example 3)

In the case of a first-price auction, the expected utility of bidder 2 from Example 3 is given by

$$U_2(x_2, b) = \int_0^{x_1(b)} [x_2 + \epsilon x_1 x_2 - b]\, f(x_1) dx_1 = F(x_1(b))(x_2 - b) + \epsilon x_2 \int_0^{x_1(b)} x_1 f(x_1) dx_1,$$

where $x_i(b)$ is the inverse bid function of player i. Differentiating U_2 with respect to b and substituting $x_2 = x_2(b)$ gives

$$x_1'(b) = \frac{F(x_1(b))}{f(x_1(b))} \frac{1}{x_2(b) + \epsilon x_1(b) x_2(b) - b}. \tag{27}$$

Repeating this procedure for bidder 1 gives

$$x_2'(b) = \frac{F(x_2(b))}{f(x_2(b))} \frac{1}{x_1(b) - b}. \tag{28}$$

The ordinary-differential equations (27,28) for the inverse equilibrium bids, together with the initial conditions $x_1(0) = x_2(0) = 0$ and the boundary condition $x_1(\bar{b}) = x_2(\bar{b})$, where $\bar{b}$ is the (unknown) maximal bid in equilibrium, are solved using a shooting method (Marshall et al. 1994). Unlike Marshall et al. (1994), we do not calculate the seller's expected revenue using Monte-Carlo methods. Rather, following Fibich and Gavious (2003), we first note that the distribution of the highest bid is given by

$$F_{1st}(b) = \Pr(\max(b_1(x_1), b_2(x_2)) \le b) = \Pr(b_1(x_1) \le b) \Pr(b_2(x_2)) \le b) = F(x_1(b)) F(x_2(b)).$$

Therefore, the seller's expected revenue is given by

$$R^{1st} = \int_0^{\bar{b}} bF'_{1st}(b)\, db = \left. bF_{1st}(b)\right|_0^{\bar{b}} - \int_0^{\bar{b}} F_{1st}(b)\, db = \bar{b} - \int_0^{\bar{b}} F(x_1(b))F(x_2(b))\, db.$$

Let us define the auxiliary equation

$$y'(b) = F(x_1(b))F(x_2(b)), \qquad y(\bar{b}) = \bar{b}. \tag{29}$$

Since $R^{1st} = y(0)$, the expected revenue is easily calculated by integrating eq. (29) backwards, once equations (27,28) have been solved.