

SUBJECTIVE DISCOUNT FUNCTIONS – AN EXPERIMENTAL APPROACH

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Abstract

This study estimates the degree of the exponential-function (EF) misvaluation and its variation with three parameters: time, the product price level and its growth rate, as well as with personal characteristics. The results suggest an undervaluation of the compound discounting formula given by the exponential function and an overvaluation of the simple-interest discounting function. Our findings appear in line with the hyperbolic function and perhaps indirectly related to the overconfidence and overreaction phenomena. A possible implication of the study's findings is that at least part of the intertemporal-choice anomalous behavior documented in the experimental literature of economic psychology can be attributed to misvaluation of the exponential function. (JEL D90, G00)

Keywords: Exponential Discount Function; Experimental Subjective Discount Rates; Linear, Quadratic and Cubic Functions.

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1. INTRODUCTION

Recent studies in financial economics and behavioral finance attempt to explain various anomalies documented in the empirical literature. In a recent extensive review study of time discounting, Frederick, Loewenstein and O'Donoghue (2002) conclude that the discount utility model has little empirical support. One of the anomalies, they note, is the declining–discount–rates result established in the economic psychology literature, and often referred to as hyperbolic time discounting. Daniel et al. (1998) summarize a large body of evidence from cognitive psychological experiments and surveys which shows that individuals overestimate their own abilities in various contexts. In a recent survey of investor psychology and asset pricing, Hirshleifer (2001) sketches a framework for understanding decision biases and discusses the importance of investor psychology.

Experimental studies of intertemporal choice derive subjective discount rates by applying the (discounting or compounding) exponential function to the sum of money in the subjects' benefit-cost responses. However, studies in psychology question the ability of subjects to evaluate the exponential function correctly. Misvaluation of the exponential function may result in anomalous subjective discount rates. One such anomaly reported in the experimental literature is the negative relationship between the time and the sum of the cash flow and the derived (implicit) subjective discount rates. This is anomalous because the impact of these two factors, time and sum, on actual capital market interest rates is generally positive.

Notwithstanding the extensive and valuable knowledge generated by the

studies reviewed below, there still seems to be a knowledge gap with respect to the exponential-function (EF) bias and its implications. The purpose of the present study therefore is threefold: (1) to estimate the degree of the EF misvaluation and its variation with a given product price level and its expected growth rate; (2) to examine whether such other mathematical functions as linear, quadratic and cubic functions, conform to the discounting and compounding process of individual decision makers; and (3) to investigate the impact of personal characteristics.¹

Using a sample of individual subjects who are familiar with the use of the exponential function discounting formula in economic and financial decision making, this experimental study finds that individuals undervalue the compound interest discounting formula given by the exponential function and overvalue the simple interest discounting formula given by the linear function (represented by Eq. 4 in the subsequent section). A comparison of four mathematical functions used in this study – the exponential, linear, quadratic and cubic - demonstrates that the degree of misvaluation is minimal for the quadratic function. At least part of the misvaluation problem can be related to the overreaction and overconfidence phenomena documented in the literature. This distortion can also be related to the “mental accounting” and “narrow framing” behavior discussed in the behavioral finance literature. A possible implication of this study is that at least part of the intertemporal-choice anomalous behavior documented in the experimental literature of economic psychology can be attributed to misvaluation of the exponential function.

The plan of the paper is as follows: Section 2 briefly reviews the relevant literature; Section 3 presents the theory, hypotheses and methodology; Section 4

outlines the experimental design; Section 5 presents the findings and discusses the results; and the last section provides a summary and concluding remarks.

2. LITERATURE REVIEW

Experimental studies in economic psychology document various types of anomalies in intertemporal choices [Thaler (1987 and 1992); Loewenstein (1988); Tversky, Slovic and Kahneman (1990) and Loewenstein and Prelec (1992)]. One of these anomalies examined by Benzion, Granot and Yagil (1992) is related to misvaluation of the exponential function which underlies the discounting and compounding of time-varying cash flows and is given by:

$$F = Pe^{kt} \quad (1)$$

where F and P are, respectively, the future and present value of the cash flow; t is the number of time periods, k is the discount (or capitalization) rate per period, and e represents the exponential function. This exponential function, it is argued in the literature, is misvalued by individuals making intertemporal choices [Wagenaar and Sagaria (1975); Wagenaar (1982); and Kemp (1987)]. Strotz (1956) is probably the first economist to consider alternatives to exponential discounting. Ainslie (1991,1992) argues that human behavior suggests that discount functions are approximately hyperbolic which is consistent with the declining-discount-rates anomaly documented in the economic psychology literature. In line with this finding, several hyperbolic functional forms have been suggested. Loewenstein and Prelec (1992) proposed the following hyperbolic form: ²

$$F = P (1+at)^{b/a} \quad (2)$$

where a and b are some constraints and P , F and t are as defined before. This discount function implies that the discount rate decreases over time. The function was found to be more consistent with observed behavior than the standard exponential function [Ainslie (1992), Loewenstein and Prelec (1992), and Kirby (1997)]. Furthermore, Laibson (1997, 1998) suggests that hyperbolic models explain a wide range of empirical anomalies and provide a framework for answering a broad range of normative questions. He demonstrates how the value of the discount function is lower for the hyperbolic than for the exponential functions in the short run, and higher in the long run. This is consistent with decreasing discount rates over time, which are implied by hyperbolic discount functions [O'Donoghue and Rabin (1999a and 1999b) and Harris and Laibson (2001)]. In his survey of 2,160 economists, Weitzman (2001) finds that the wide spread of opinion on what the social discount rate should be make the social discount rate decline significantly over time. In their study of the military downsizing program of the early 1990's, Warner and Pleeter (2001) find that the estimates of the personal discount rate range from 0 to over 30 percent, and vary with personal characteristics.

As noted above, the exponential-function misvaluation seems to be related to the overconfidence and overreaction phenomena reexamined recently in the literature. Based on the premise of investor overconfidence, Daniel et al. (1998) note two well-known psychological biases: investor overconfidence about the precision of information and biased self-attribution, which causes asymmetric shifts in investor confidence. The overconfidence phenomenon has been recently investigated further by Barber and Odean (2000) and Gervais and Odean (2001).

Barberis, Shleifer and Vishny (1998) also note the large body of evidence concerning underreaction and overreaction, according to which security prices underreact to news in the short run, but overreact to consistent patterns of news pointing in the same direction in the long run. Their findings, they emphasize, challenge the efficient market theory. In a recent study on mental accounting, Barberis and Huang (2001) argue that it is possible to improve our understanding of firm-level stock returns by employing the experimental evidence related to Kahneman and Tversky's (1979) "loss aversion" concept, according to which people are more sensitive to losses than to gains. This concept is closely related to Thaler's (1987) "mental accounting" term, which refers to the process by which people think about and evaluate their financial transactions. Experimental studies suggest that when doing their mental accounting, people engage in "narrow framing" which refers to narrowly defined gains and losses. Loss aversion and narrow framing have already been applied to aggregate stock market and to retirement investment by Benartzi and Thaler (1995, 1999). Motivated by their work, Barberis, Huang and Santos (2001) introduce loss-aversion over financial wealth fluctuations into a dynamic equilibrium model. The loss-aversion concept has been recently advanced by Loewenstein (2000) and Rabin and Thaler (2001).

3. THEORY, HYPOTHESES AND METHODOLOGY

The exponential function describes a continuous compound interest relation given by Eq. (1) in the preceding section. For discrete (rather than continuous) compounding, Eq. (1) reduces to

$$F = P(1 + g)^t; (F/P) = (1 + g)^t; (F/P) = a + b(1 + g)^t \quad (3)$$

where, as defined before, F is the future value of a cash flow; P is the present value; t is the number of time periods; and g is the growth (or interest) rate and corresponds to the discount rate (k) in Eq. (1). The third expression in Eq. (3) represents the regression form, where, by the null hypothesis, a = 0 and b = 1. This regression equation will be estimated in order to test the degree to which individuals underestimate the exponential function. This degree will be represented by how far is the b coefficient in Eq. (3) different than unity.

Despite a potential misvaluation of the exponential function, individuals who are aware of the compounding process do not just base their estimate merely on the linear function underlying the “simple” interest relation, but probably add a “compounding premium” to the linear-function estimate. By the linear-function relation,

$$F = P(1 + gt); F/P = 1 + gt; F/P = a + bgt, \quad (4)$$

where all symbols are defined as before. Eq. (4) is in fact a reduced form of Mazur’s (1987) version of the type of discount functions documented in prior parametric studies of behavioral psychologists. The third expression in Eq. (4) represents the regression form, where, by the null hypothesis, a = b = 1. However, the existence of a “compounding premium” will make the slope greater than unity. Such a slope value implies that individuals are aware of the exponential function but do not comprehend its full impact, especially for high values of time and growth rate. In such cases, they employ the linear-function approximation and then add a

“compounding premium” accounting for the compounding process underlying the exponential function. This premium is analogous to De Bondt’s (1993) hedging argument in his study of non-experts’ intuitive forecasts of financial risk and return.

To investigate the extent to which subjects’ responses conform to mathematical functions other than the exponential or the linear, two additional functions - the quadratic and cubic, which follow from Taylor series expansions - will be applied to the exponential growth function given in Eq. (3). That is,

$$(1+g)^t = 1 + tg + [t(t-1)/2]g^2 + [t(t-1)(t-2)/6]g^3 + \dots \quad (5)$$

$$F = P(1+g)^t = P[1+tg+[t(t-1)/2]g^2 + [t(t-1)(t-2)/6]g^3] \quad (6)$$

where the left side of Eq. (5) and the first two, three and four terms on the right side of Eq. (5) represent the exponential, linear, quadratic and cubic functions, respectively. In Eq. (5), $(1+g)^t$ is simply the price ratio given by the (F/P) ratio, where F, to be recalled, is the future value and P is the present value. Hence, Eq. (6) is simply the dollar equivalent of Eq. (5). It is thus possible to compare individuals’ subjective F/P ratio (SR) with the computed ratio (CR) given by the four functions.³ Specifically,

$$CRE = (1+g)^t \quad (7A)$$

$$CRL = 1 + tg \quad (7B)$$

$$CRQ = 1 + tg + [t(t-1)/2]g^2 \quad (7C)$$

$$CRC = 1 + tg + [t(t-1)/2]g^2 + [t(t-1)(t-2)/6]g^3 \quad (7D)$$

where CRE, CRL, CRQ and CRC denote the computed price ratio (CR) by the exponential, linear, quadratic and cubic functions, respectively. As in Eq. (5), multiplying the right side of Eq. (7) by P yields F or, equivalently, the computed price (CP) on the left side of these equations. For the four functions, the computed price will be denoted by CPE, CPL, CPQ and CPC, respectively. For example, CPE=F= P (1+g)^t.

To examine which of the above four functions best fits the subjective ratio (SR), the following regression equation will be estimated:

$$SR_{t,i} = a + b(CR)_t + e_{t,i} \quad (8)$$

where t = 1,2,..., T time periods; i = 1,2,...,I subjects; e is the error term; and the number of observations (n) will be given by the product TI. Equation (8) will be estimated n times corresponding to n growth rates, which will be exogenously given. This set of n equations will be estimated four times corresponding to the four types of computed ratios: CRE, CRL, CRQ and CRC. The null hypothesis by the exponential function is: a=0 and b=1, and by the linear function: a=b=1.

For the quadratic and cubic functions, however, we only know that a=0 and b should be greater than unity. The following two regression forms of Eqs (7C) and (7D) give a more direct test of these two functions.

$$SR_{t,i} = a + btg_t + c[t(t-1)/2]g_t^2 + e_{t,i} \quad (9)$$

$$SR_{t,i} = a+btg_t + c[t(t-1)/2]g_t^2 + d[t(t-1)(t-2)/6]g_t^3 + e_{t,i} \quad (10)$$

The null hypothesis for both equations is: $a = b = c = d = 1.0$. Note that in these two equations the quadratic and cubic functions are represented *explicitly*, whereas in Eq. (8), they are only represented implicitly.

For each hypothetical product in the questionnaire, represented by a given present price (GP) and a given growth rate (GG), subjects in the experiment are required to provide their estimate for the subjective price (SP) of item j at time t . To establish a price misvaluation, this vector of subjective prices is compared with the computed price (CP) given by the (compounding) exponential function (Eq. 3), and the (simple interest) linear-function price given by Eq. (4). As a coefficient, the price misvaluation (PM) is defined as:

$$PM = (SP/CP) \quad (11)$$

For each subject, we can define a subjective misvaluation coefficient (SMC) across various scenarios. That is

$$SMC = (1/Q) \sum_q (PM_q - 1); q = 1, 2, \dots Q \text{ scenarios} \quad (12)$$

Equivalently, the exponential-function bias can be represented by the gap between the given growth rate (GG) and the subjective growth rate (SG) derived from the subjective price (SP). Formally,

$$SG = (SP/GP)^{1/t} - 1 \quad (13)$$

where GP is the given price. A misvaluation of the exponential function implies a price misvaluation (PM) coefficient lower than unity; i.e. a subjective growth rate (SG) that is lower than the given growth rate (GG).

One way to test whether the magnitude of undervaluing the exponential function is greater than that associated with overvaluing the linear function is to compute the price ratio dispersion (PRD), defined as the standard deviation of the gap between the subjective price ratio (SR) and the computed price ratio (CR). In addition, the quadratic and cubic functions also will be incorporated. Formally:

$$PRD_i = [(1/n) \sum_t \sum_g (SR_{i,t,g} - CR_{t,g})^2]^{1/2} \quad (14)$$

For the four functions, Eq. (14) will be:

$$PRDE_i = [(1/n) \sum_t \sum_g (SR_{i,t,g} - CRE_{t,g})^2]^{1/2} \quad (14A)$$

$$PRDL_i = [(1/n) \sum_t \sum_g (SR_{i,t,g} - CRL_{t,g})^2]^{1/2} \quad (14B)$$

$$PRDQ_i = [(1/n) \sum_t \sum_g (SR_{i,t,g} - CRQ_{t,g})^2]^{1/2} \quad (14C)$$

$$PRDC_i = [(1/n) \sum_t \sum_g (SR_{i,t,g} - CRC_{t,g})^2]^{1/2} \quad (14D)$$

where $i=1,2,\dots, I$ subjects; $t=1,2,\dots, T$ time periods; $g=1,2,\dots, G$ growth rates; $n=TG$; and PRDE, PRDL, PRDQ and PRDC are the price ratio dispersion between the subjective price ratio (SR) and the computed price ratio (CR), respectively, for the exponential, linear, quadratic and cubic functions.

If individuals undervalue the exponential function and overvalue the linear function, it would be important to investigate whether the discounting and compounding process of individual decision making is described better by the quadratic and cubic functions than the exponential and linear functions. Finally, Eq. (12) is employed to investigate the impact of basic personal characteristics on the misvaluation phenomenon.

4. THE EXPERIMENTAL DESIGN

The questionnaire, fully presented in the appendix, consists of two parts: In the first part, subjects were asked to state their prices in the past (two time periods) and in the future (six time periods) for five hypothetical items (goods) given their current prices and the annual price change for each item. The research purpose in this part is to evaluate the degree to which the subjects' responses are consistent with the exponential, linear, quadratic or cubic functions. In the second part of the questionnaire, subjects were asked several questions relating to such personal characteristics as sex, age, having a savings account and investing in capital-market securities.

The questionnaire was distributed to 186 economics students at the University of Pennsylvania. Of the 186 subjects who participated in the experiment, 23 did not complete the questionnaire and were omitted from the sample. Consequently, the results reported below are based on a sample of 163.

The sample was drawn from the classes of one of the authors. Though the subjects were not paid, we believe they seriously considered and answered the questions. In this respect, Loewenstein (1999) argues that experimental economists

should not deceive themselves into believing that the use of such rewards allows them to control the incentives operating in their experiments. No instructions were given other than those in the questionnaire. One of the instructions in the questionnaire asks the subjects to evaluate the set of prices using no calculators. Though in real life calculators and PC programs are available, the emphasis in this study is the behavioral aspects of individual decision makers' attitudes, perceptions and evaluation of the impact of time, monetary value and growth rates.

The advantages and disadvantages of experimental studies noted in the literature apply to this study, too. The questionnaires were examined, and no notable differences in response pattern were detected. Furthermore, as discussed later, the results found are consistent with the hyperbolic function established in the literature. In addition, personal characteristics are shown later to be of no significant consequence. Specifically, no evidence is found for sample segmentation, whereby subjective responses in one subsample, conform to one type of discount function, and responses in another subsample conform to another type of discount function.

The questionnaires were distributed during class time for students enrolled in one of the authors' courses. Students were given sufficient response time and were informed of the relevance of the topic. They were also informed that the findings of the experiment would be discussed in a future class. Therefore, it would seem that the students were positively motivated to complete the project with the necessary care and diligence, even without being offered monetary incentives - which, psychologists claim, do not necessarily improve performance. Gneezy and Rustichini's (2000) experimental findings are consistent with this claim though, in their prisoner's dilemma classroom game, Holt and Capra (2000) find that the extent

of the cooperation is often affected by the payoff incentives and by the nature of repeated interaction.

5. RESULTS

5.1 Descriptive Analysis

For the five items, represented by the exogenously given annual growth rates (GG) and present price (GP), Table 1 presents the subjective prices (SP) in the future and in the past based on subjects' responses. In addition to the mean subjective price, the standard deviation (SDV) across subjects, as well as the coefficient of variation (COV) defined as $SDV/MEAN$, are computed. The results indicate that the COV increases with time. Furthermore, for larger t values, it also increases with the growth rate of the item's price.

The subjective as well as the computed price ratios given by Eq. (7) are presented in Table 2 and graphically in Exhibits 2-A and 2-B.

Exhibit 2-A depicts the relationship between the price ratio and the growth rate and is based on the Mean column in Table 2.

Exhibit 2-B, in comparison, depicts the relationship between the price ratio and time, as reflected in the Mean row in Table 2. Both the table and the two exhibits demonstrate that for relatively low and moderate values of t and g (up to 8 years and 20%, respectively), the subjective price ratio is close to the computed ratio, particularly by the quadratic and linear functions. For extreme values of t and g (20 years and 50%, respectively), however, the quadratic function provides the best fit, whereas the linear and exponential functions are deeply overvalued and undervalued, respectively.

To investigate these differences more explicitly, the price misvaluation (PM) coefficient, given in Eq. (11) as the ratio between the subjective price and the computed price, was calculated for each of the four functions across the growth and time values. The results are presented in Table 3 and Exhibits 3-A and 3-B. The lowest overall gap between the functions is demonstrated in the part of the Mean column in Table 3. The lowest overall (across t and g) mean value of PM was found for the exponential function (0.79) and highest for the linear function (1.36). For the quadratic and cubic functions the mean PM is about 0.9 and 0.8, respectively. The reasonable fit of the quadratic compared with the other functions is also depicted in Exhibits 3-A and 3-B as a function of t and g , respectively.

Consistent with the exponential undervaluation phenomenon indicated by Table 2, the price misvaluation (PM) in Table 3 is generally less than unity. More notably, the findings in Table 3 suggest the following: (1) PM generally decreases with time irrespective of the level of the growth rate; (2) it decreases with the growth rate irrespective of time; (3) these two results imply that PM reaches extremely low values for extremely high values of time and growth rate. These results are also seen in the subjective growth rate (SG) given by Eq. (13) and reported in Table 4. SG is defined as $(SP/GP)^{1/t} - 1$, where GP is the given present price, SP is the subjective price, and t is the number of years. The subjective growth rate (SG), as demonstrated by Table 4, generally decreases with time. More specifically, for short periods (forwards or backwards) the subjective growth rate is generally greater than the given growth rate (GG), whereas for longer forward years, it is lower.

To characterize each individual subject as undervaluing or overvaluing the exponential function, the price misvaluation (PM), given by Eq. (11) was computed

for each subject across his responses to the four questions. This “personal” PM was called subjective misvaluation coefficient (SMC) and is computed by Eq. (12). The mean and standard deviation of SMC were then averaged across the 163 subjects in the sample. The resulting mean SMC value was -0.12 with a standard deviation of 0.31 and the maximum and minimum values were 2.48 and -0.77, respectively. Of the 163 subjects, 93 percent had a negative SMC value; meaning they undervalued the exponential function.

To investigate whether the magnitude of undervaluing the exponential function is greater than that associated with overvaluing the linear function, and in relation to the quadratic and cubic functions, the price dispersion ratio (PRD) was computed. As defined earlier, it is given by the standard deviation of the gap between the subjective price ratio (SR) and the computed price ratio (CR) given by Eq. (14), and the results appear in Table 5. The number of observations underlying the PRD is the product of eight time periods and five growth rates. The ratio PRD is computed for each of the 163 subjects and for each of the four functions. The mean value (across the subjects) and its coefficient of variation (COV), defined as the mean over its standard deviation, are presented in Table 5 for the eight time periods examined. The results demonstrate that for backward years (-5 and -1) and short forward years (up to +2), the exponential price dispersion (PRDE) is not statistically different than the linear price dispersion (PRDL). However, for distant forward years (+5 to +20), PRDE is much larger, implying a greater misvaluation of the exponential than the linear function. Also, for distant forward years, the coefficient of variation (COV) associated with the price ratio dispersion decreases with time for the exponential function and increases with time for the linear function. This last result

implies that for long time periods, the misvaluation degree of the exponential function is very high, overshadowing personal differences. As demonstrated in Table 5, similar results are obtained for the variation of the price ratio dispersion with the growth rate (rather than time).

In relation to the additional two functions examined – the quadratic and the cubic - the price ratio dispersion for the quadratic was found to be lower than for the cubic. This effect of continued error in estimation represents what Brandts and Holt (1995) call the limitation of dominance and forward induction. For both functions, however, the price ratio dispersion lies between the maximum value obtained for the exponential and the minimum value obtained for the linear functions.

5.2 Regression Analysis

To directly test the magnitude of the undervaluation of the exponential function, the regression form of the exponential function given by Eq. (3) was estimated five times for each of the five products (or growth rates) given in the questionnaire, and for the six forward time periods given. The results, presented in the upper part of Table 6, indicate that, the slope coefficient (b) is less than unity and statistically significant at 1%. Moreover, it decreases sharply with the growth rate (g); it amounts to 0.81 for g of 3% and only 0.02 for g of 50%. This reduction in the regression slope (b) is also associated with a corresponding reduction in R^2 from 0.48 for $g = 3\%$ to only 0.09 for $g = 50\%$. All five-regression equations, however, are significant at 1% as indicated by their F-values. These results imply an undervaluation of the exponential function, which is moderate for low growth or interest rates and substantial for high growth rates.

The “compounding premium” postulated in the previous section suggests that individuals use the simple-interest linear function and, to account for the compounding process, add a “compounding premium.” According to this hypothesis, the slope coefficient of the regression form of the linear function given by Eq. (4) should be greater than unity. As in the previous exponential function’s case, Eq. (4) was estimated five times for each of the five growth rates, and the results appear in the lower part of Table 6. The results indicate that the slope coefficient (b) is always greater than unity and statistically significant at 1%. Moreover, it increases from 1.1 for a low growth rate of 3% to 7.8 for a high growth rate of 50%. As expected, R^2 , in contrast, decreases from 0.48 for $g = 3\%$ to 0.08 for $g = 50\%$, but all five equations are significant at 1%. The reduction in R^2 implies that for very high g values, the linear function becomes only a crude estimate of the compound value and a heavier weight is attached to the “compounding premium,” probably imbedded in the constant term of the regression. These results imply that the linear function is used as a proxy for the compounding process and an adjustment is made by incorporating a compounding premium.

Similarly, the slope coefficient (b) for the quadratic and cubic functions too is lower than unity in all regressions estimated in Table 7. The b values, however, especially for the quadratic function, are closer to unity though statistically different than unity. These results indicate the same undervaluation phenomenon reported for the exponential function, though it is more moderate.

A more explicit test of the quadratic and cubic functions is provided by Eqs. (9) and (10), respectively. The results are presented in Table 8. Based on the two null hypotheses, the two slope coefficients in the quadratic function (Eq. 10) are

expected to be equal to unity. An inspection of Table 8 reveals that most of these slope coefficients are not statistically different than unity.

5.3 The Effect of Personal Characteristics

Four personal characteristics were examined for the sample; sex, age, having a savings account and investing in capital market securities. The descriptive statistics indicate that 68.5% of the subjects are male, their mean age is 20 years; 84.7% have a savings account; and 24.5% invest in capital market securities. Correlation results show that compared to female students, male students invest more in securities and that having a savings account is positively correlated with investing in capital market securities.

Other correlations between the four personal characteristics were not statistically significant. The effect of personal characteristics on the subjective misvaluation coefficient (SMC) also was examined, where SMC, given by Eq. (12), indicates for each subject the gap between the subjective and the computed prices across the forty questions asked. Regression results show no statistically significant correlations between SMC and personal characteristics.

6. SUMMARY AND CONCLUDING REMARKS

The objective of this study has been threefold: (1) to estimate the degree of the exponential-function (EF) misvaluation and its variation with the product price level and its expected growth rate; (2) to examine whether other mathematical functions, such as the linear, quadratic and cubic functions conform to the discounting and compounding process of individual decision makers; and (3) to

investigate the impact of personal characteristics.

The sample selected to test the hypotheses postulated in this study consists of individual subjects who are familiar with the use of the exponential function discounting formula in economic and financial decision-making.

The results indicate an undervaluation of the compound interest discounting formula given by the exponential function, and an overvaluation of the simple-interest discounting formula given by the linear function.

The findings concerning the price misvaluation (PM coefficient), defined as the subjective price (SP) over the exponential-function computed price (CP), demonstrate more sharply the undervaluation phenomenon. Specifically, PM increases with time (t) irrespective of the growth rate (g) and increases with g irrespective of t. Furthermore, for backward years and short forward years (up to +2), the exponential price dispersion (between the subjective and computed prices) is lower than the linear price dispersion, and for distant forward years (+2 to +20) it is higher. Regression results also indicate that the linear function is used as a proxy for the compounding process, and a compounding premium is added for adjustment.

A comparison of the four mathematical functions used – the exponential, linear, quadratic and cubic - with respect to the price misvaluation (PM) coefficient demonstrates that for relatively moderate values of time (t) and growth rate (g), the degree of misvaluation is relatively low, especially for the linear and quadratic functions. For large values of t and g, the misvaluation degree is high, but is minimal for the quadratic function. These results are also confirmed by the regression analysis. Of the four functions tested, the quadratic function had the lowest gap between the estimated OLS coefficients and their theoretical counterparts.

Finally, regression results demonstrate no statistically significant correlation between the exponential function (EF) subjective misvaluation coefficient and such personal characteristics, as sex, age, having a savings account and investing in capital market securities. This result implies that the EF misvaluation is robust; personal differences are of no significant consequence.

Our findings appear in line with the hyperbolic function that was found consistent with observed behavior and can explain a wide range of empirical anomalies (Laibson 1997, 1998). Falling discount rates, implied by the hyperbolic function, suggest that the future value of a given sum will be lower than that given by the exponential function, which usually assumes constant discount rates.

One implication of our findings is that when individuals misvalue the exponential function and behave in line with the hyperbolic function (which implies higher discount rates in the short run and lower in the long run), they will reach different consumption and saving decisions, as noted already by Laibson (1997,1998). At least part of the misvaluation problem may be attributed to the overconfidence phenomenon investigated by Daniel et al. (1998) and, more recently, by Gervais and Odean (2001), and to the overreaction phenomenon in the long run noted by Barberis, Shleifer and Vishny (1998). The exponential misvaluation phenomenon, which yields results similar to those implied by the hyperbolic function, can distort individuals' decisions between the short and long run, as correctly noted before in the literature. This distortion can also be related to the "mental accounting" and "narrow framing" documented in behavioral finance and reexamined recently by Barberis et al. (2001) and Barberis and Huang (2001), who note that the discount rate behavior is the key to many of the portfolio accounting results. In a recent

working paper entitled “Can the Market Add and Subtract...” Lamont and Thaler (2001) argue that market participants should quickly destroy any discrepancies in valuations, but this does not always happen, they conclude. Perhaps, the misvaluation problem investigated here should be considered in the broader context of the link between emotions and economics discussed by Elster (1998) and Loewenstein (2000), and the relationship between psychology and economics surveyed by Rabin (1998).

Finally, it might be worth emphasizing that a possible implication of this study is that at least part of the intertemporal-choice anomalous behavior documented in the experimental literature of economic psychology can be attributed to subjective misvaluation of the compounding and discounting process represented by the exponential function.

Footnotes

1. In this respect, the objective is to gain a better understanding of what Loomes (1999) calls “the disparate and context-dependent ways in which people handle decision problems,” in his analysis of what experimental economics can achieve.
2. Other hyperbolic functional forms suggested in the literature include the following: Ainslie (1975): $F = Pt^{-1}$ and Herrnstein (1981) and Mazur (1987): $F = P(1 + at)^{-1}$, where F , P and t are as defined above, and a is some empirical constant.
3. Due to the relatively large number of symbols, a notation summary is provided in the appendix, organized by the order of the symbols’ appearance in this paper.

Appendix

Notations:

EF= exponential function

F= future value

P= present value

t= time

k= discount rate

g= growth or interest rate

SR= subjective ratio of (F/P)

CR= computed ratio of (F/P)

CRE; CRL; CRQ; CRC= computed ratio (CR) by the exponential, linear, quadratic and cubic functions, respectively.

CP= computed price

CPE; CPL; CPQ; CPC= the computed price by each of the four functions

GP= given present price

SP= subjective price

GG= given growth rate

SG=subjective growth rate

PM= price misvaluation given by (SP/CP)

SMC= subjective misvaluation coefficient

PRD= pride ratio dispersion

PRDE; PRDL; PRDQ; PRDC= the price ratio dispersion (PRD) for the four functions.

COV= coefficient of variation.

REFERENCES

- Ainslie, G. (1975), "Specious Rewards: A Behavioral Theory of Impulsiveness and Impulse Control," *Psychological Bulletin* 82(4), 463-496.
- Ainslie, G. (1991), "Derivation of Rational Economic Behavior from Hyperbolic Discount Curves," *American Economic Review* 81 (2), May, 334-340.
- Ainslie, G. W. (1992), *Picoeconomics*. Cambridge University Press, Cambridge.
- Barber, B. M. and T. Odean (2001), "Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors," *The Journal of Finance* 55 (2), April, 773-806.
- Barber, B. M. and T. Odean (2000), "Boys Will Be Boys: Gender, Overconfidence, and Common Stock Investment," *Quarterly Journal of Economics* 115 (1), February, 261-292.
- Barberis, N., A. Shleifer, and R. Vishny (1998), "A Model of Investor Sentiment," *Journal of Financial Economics* 49, 307-343.
- Barberis, N. and M. Huang (2001), "Mental Accounting, Loss Aversion and Individual Stock Returns," *NBER working paper No. W1890*, March.
- Barberis, N., M. Huang, and T. Santos (2001), "Prospect Theory and Asset Prices," *Quarterly Journal of Economics*, 116, 1-53.
- Benartzi, S. and R. Thaler (1995), "Myopic Loss Aversion and the Equity Premium Puzzle," *Quarterly Journal of Economics* 110, 73-92.
- Benartzi S. and R. Thaler (1999), "Risk Aversion or Myopia? Choices in Repeated Gambles and Retirement Investments," *Management Science* 45, 364-381.
- Benzion, U., A. Granot, and J. Yagil (1992), "The Valuation of the Exponential Function and Implications for Derived Interest Rates," *Economics Letters* 38,

March, 299-303.

Brandts, J. and C. Holt (1995), "Limitations of Dominance and Forward Induction: Experimental Evidence," *Economics Letters* 49 (4), October, 391-395.

Daniel, K., D. Hirshleifer and A. Subrahmanyam (1998), "Investor Psychology and Security Market Under- and Overreactions," *Journal of Finance* 53(6), December, 1839-85.

De Bondt, F. (1993), "Betting on Trends: Intuitive Forecasts of Financial Risk and Return," *International Journal of Forecasting* 9, 355-371.

Elster, J. (1998), "Emotions and Economic Theory," *Journal of Economic Literature* 86, March, 47-74.

Fredrick, S., G. Loewenstein and T. O'Donoghue (forthcoming), "Time Discounting: A Critical Review," *Journal of Economic Literature*.

Gervais, S. and T. Odean (2001), "Learning to Be Overconfident," *Review of Financial Studies*, 14(1), Spring, 1-27.

Gneezy, U. and A. Rustichini (2000), "Pay Enough or Don't Pay At All," *Quarterly Journal of Economics* 155 (3), August, 791-810.

Harris, C. and D. Laibson (2001), "Dynamic Choices of Hyperbolic Consumers," *Econometrica* 69(4), July, 935-957.

Herrnstein, R. (1981), "Self-Control as Response Strength," in C.M. Bradshaw et al. eds., *Quantification of Steady-State Operant Behavior*, Elsevier/North-Holland.

Hirshleifer, D. (2001), "Investor Psychology and Asset Pricing," *Journal of Finance*, 56 (4), August, 1533-1597.

Holt, C. and M. Capra (2000), "Classroom Games: A Prisoner's Dilemma," *Journal of Economic Education* 31(3), Summer, 229-236.

- Kahneman, D. and A. Tversky (1979), "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica* 47, 359-363.
- Kemp, S. (1987), "Estimating Past Prices," *Journal of Economic Psychology* 8, 181-9
- Kirby, K. (1997), "Bidding on the Future: Evidence Against Normative Discounting of Delayed Rewards," *Journal of Experimental Psychology* 126 (2), 57-71.
- Laibson, D. (1997), "Golden Eggs and Hyperbolic Discounting," *The Quarterly Journal of Economics* 112 (2), May, 443-447.
- Laibson, D. (1998), "Life-Cycle Consumption and Hyperbolic Discount Functions," *European Economic Review*, 42 (3), May, 861-871.
- Lamont, O. and R. Thaler (2001), "Can the Market Add and Subtract: Mis-pricing in Tech Stock Carve-outs," <http://www.uchicago.edu/fac/finance/papers/>
- Loewenstein, G. (1988), "Frames of Mind in Intertemporal Choice," *Management Science* 34, 200-214.
- Loewenstein, G. (1999), "Experimental Economics from the Vantage-Point of Behavioral Economics," *The Economic Journal*, February.
- Loewenstein, G. (2000), "Emotions in Economic Theory and Economic Behavior," *American Economic Review* 90 (2), May, 420-432.
- Loewenstein, G. and D. Prelec (1992), "Anomalies in Intertemporal Choice: Evidence and an Interpretation," *Quarterly Journal of Economics* 57 (2), 573-598.
- Loomes, G. (1999), "Some Lessons from Past Experiments and Some Challenges for the Future," *The Economic Journal* 45, 35-45.
- Mazur, J. E. (1987), "An Adjusting Procedure for Studying Delayed Reinforcement," in M. L. Commons et al., eds., *The Effect of Delay and Intervening Events on*

Reinforcement Value, Hillsdale, NJ, Erlbaum.

O'Donoghue, T. and M. Rabin (1999a), "Doing It Now Or Later," *American Economic Review* 89, 103-124.

O'Donoghue, T. and M. Rabin (1999b), "Incentives For Procrastinators," *Quarterly Journal Of Economics* 114, 769-816.

Rabin, M. (1998), "Psychology and Economics," *Journal of Economic Literature* 36, March, 11-46.

Strotz, R. H. (1956), "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies* 23, 165-180.

Thaler, R. H. (1987), "Anomalies: Saving, Fungibility and Mental Accounts," *Journal of Economic Perspective* 1, Winter, 197-201.

Thaler, R. H. (1992), *The Winner's Curse: Paradoxes and Anomalies of Economic Life*, New York: The Free Press.

Tversky, A., P. Slovic, and D. Kahneman (1990), "The Causes of Preference Reversal," *American Economic Review* 80, 204-217.

Wagenaar, W. A. (1982), "Misperception of Exponential Growth and the Psychological Magnitude of Numbers," in B. Wegener (Ed.), *Social Attitudes and Psychological Measurements*, Erlbaum, Hillsdale, N.J., 282-301.

Wagenaar, W. and S. Sagaria (1975), "Misperception of Exponential Growth," *Perception and Psychophysics* 18, 416-422.

Warner, J.T. and S. Pleeter (2001), "The Personal Discount Rate: Evidence from Military Downsizing Programs," *American Economic Review* 91, March, 33-53

Weitzman, M.L. (2001), "Gamma Discounting," *American Economic Review* 91(1), March, 260-271.

Questionnaire on Problems in Financial Economics

The purpose of the present experiment is to obtain some estimates of your evaluations for prices of hypothetical goods in the past and in the future based on given price changes typically associated with them. The information gathered will be used for research purposes only. Please answer all questions. Due to methodological reasons incomplete questionnaires may be omitted. We appreciate the time you take to answer this questionnaire, and we thank you for your cooperation.

In the following table you are given the current prices of five items and their percentage annual average price changes in the past, which are also expected to prevail in the future. That is, the price change is constant over time but it differs for different products. Based on the current price for each item given below, and the annual price change associated with it, please evaluate (using no calculators) the item's price for the different points in time in the past and in the future stated in the table below. Please round your answer to the nearest dollar.

Item	Annual Price Change (%)	ESTIMATED PRICES (\$)								
		5 Years ago	1 Year ago	Today's Price	1 Year Hence	2 Years Hence	5 Years Hence	8 Years Hence	12 Years Hence	20 Years Hence
A	3			90						
B	7			30						
C	10			50						
D	20			70						
E	50			20						

Personal characteristics: Circle your choice

Sex: M F; Age ____; Do you have a savings account? Yes No ; Do you invest in capital - market securities? Yes No . Thanks again and good luck in your program.

Table 1**The Mean (across Subjects) Subjective Price (SP) by Time and Growth Rate for Five Hypothetical Items (\$)**

Item	g (%)	5P	-5	-1	0	1	2	5	8	12	20
A	3	mean	76.8	86.5	90.0	93.4	96.3	103.7	113.0	126.3	149.4
		Sdv	9.0	3.3		3.5	3.5	7.8	16.8	23.4	39.1
		cov	0.1	0.0		0.0	0.0	0.1	0.1	0.2	0.3
B	7	mean	19.0	26.1	30.0	33.3	37.0	44.8	53.7	64.3	83.5
		sdv	5.8	3.1		2.3	4.6	9.3	14.5	23.8	38.2
		cov	0.3	0.1		0.1	0.1	0.2	0.3	0.4	0.5
C	10	mean	31.6	44.6	50.0	55.2	61.1	74.4	89.2	108.8	176.1
		sdv	9.7	2.2		2.2	4.4	9.6	17.8	31.0	192.2
		cov	0.3	0.0		0.0	0.1	0.1	0.2	0.3	1.1
D	20	mean	36.8	56.9	70.0	82.6	96.7	126.9	164.7	216.0	413.0
		sdv	14.2	7.2		5.4	10.4	30.0	59.4	121.2	656.0
		cov	0.4	0.1		0.1	0.1	0.2	0.4	0.6	1.6
E	50	mean	5.8	12.4	20.0	30.4	456.0	107.0	198.1	437.3	1575.8
		sdv	5.0	4.5		4.6	11.0	146.1	232.6	696.8	4034.7
		cov	0.9	0.4		0.2	0.2	1.4	1.2	1.6	2.6

Notation: $g \neq GG$ = given growth rate; SP = the subjective price

The time variable (years) is given in the first line of the table, where "0" is the present-time price of the five items as given to the subjects; past and future are denoted by minus and plus, respectively. The sample size is 163 subjects.

Table 2
The Mean (across Subjects) Subjective Price Ratio (SR) and the Computed Price Ratio (CR) by the Exponential Function (CRE), the Linear Function (CRL) the Quadratic Function (CRQ) and the Cubic Function (CRC) by Time and Growth Rate for Five Hypothetical Items

Item	g(%)	PR	-5	-1	0	1	2	5	8	12	20	Mean
A	3	SR	0.85	0.96	1.0	1.04	1.07	1.15	1.26	1.40	1.66	1.26
		CRE	0.86	0.97		1.03	1.06	1.16	1.27	1.43	1.81	1.29
		CRL	0.87	0.97		1.03	1.06	1.15	1.24	1.36	1.60	1.24
		CRQ	0.86	0.97		1.03	1.06	1.16	1.27	1.42	1.78	1.29
		CRC	0.86	0.97		1.03	1.06	1.16	1.27	1.43	1.82	1.30
B	7	SR	0.63	0.87	1.0	1.11	1.23	1.49	1.79	2.14	2.78	1.76
		CRE	0.71	0.93		1.07	1.14	1.40	1.72	2.25	3.87	1.91
		CRL	0.74	0.93		1.07	1.14	1.35	1.56	1.84	2.40	1.56
		CRQ	0.71	0.93		1.07	1.15	1.41	1.72	2.19	3.38	1.82
		CRC	0.71	0.93		1.07	1.15	1.42	1.75	2.29	3.84	1.92
C	10	SR	0.63	0.89	1.0	1.10	1.22	1.49	1.78	2.18	3.52	1.88
		CRE	0.62	0.91		1.10	1.21	1.61	2.14	3.14	6.73	2.66
		CRL	0.67	0.91		1.10	1.20	1.50	1.80	2.20	3.00	1.80
		CRQ	0.62	0.92		1.11	1.22	1.63	2.12	2.92	5.00	2.33
		CRC	0.61	0.91		1.11	1.22	1.65	2.21	3.21	6.33	2.62
D	20	SR	0.53	0.81	1.0	1.18	1.38	1.81	2.35	3.09	5.90	2.62
		CRE	0.40	0.83		1.20	1.44	2.49	4.30	8.92	38.34	9.45
		CRL	0.50	0.83		1.20	1.40	2.00	2.60	3.40	5.00	2.60
		CRQ	0.40	0.83		1.22	1.48	2.50	3.88	6.28	13.00	4.73
		CRC	0.38	0.83		1.22	1.49	2.67	4.56	8.58	23.67	7.03
E	50	SR	0.29	0.62	1.0	1.52	2.30	5.35	9.91	21.86	78.79	19.96
		CRE	0.13	0.67		1.50	2.25	7.59	25.63	129.	3325.2	582.00
		CRL	0.29	0.67		1.50	2.00	3.50	5.00	7.00	11.00	5.00
		CRQ	0.15	0.67		1.63	2.50	6.63	13.00	25.00	61.00	18.29
		CRC	0.11	0.67		1.65	2.67	9.23	23.67	61.00	227.67	54.31
MEAN		SR	0.59	0.83		1.19	1.44	2.26	3.42	6.13	18.53	3.93
		CRE	0.55	0.86		1.18	1.42	2.85	7.01	29.10	675.20	79.91
		CRL	0.61	0.86		1.18	1.36	1.90	2.44	3.16	4.60	1.90
		CRQ	0.55	0.86		1.21	1.48	2.66	4.40	7.56	16.83	4.06
		CRC	0.53	0.86		1.22	1.52	3.22	6.69	15.3	52.66	9.22

Notation: $g \neq GG$ = given growth rate; PR = price ratio; $PR = F/P$, where F is future value and P is present value. The time variable (years) is given in the first line of the Table, where "0" is the present price of the five items as given to the subjects; past and future are denoted by minus and plus respectively; $CRQ = 1+tg + [t(t-2)/2]g^2$; $CRE = (1+g)^t$; $CRL = 1+tg$; $CRC = 1+tg + [t(t-1)/2]g^2 + [t(t-1)(t-2)/6]g^3$. These equations hold for $t > 0$. For $t < 0$, the inverse price ratio, namely (P/F), should be used. The mean column is across the forward years only.

Exhibit 2-A
 The Subjective Price Ratio (PR) and the Computed PR
 by the Four Discounting Functions for Different Growth Rates

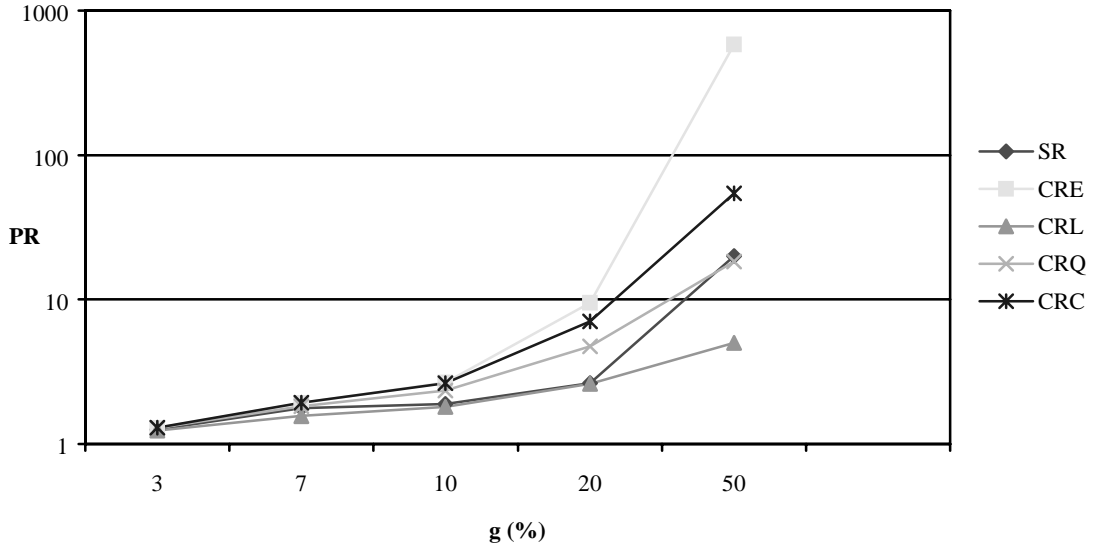


Exhibit 2-B
 The Subjective Price Ratio (PR) and the Computed PR
 by the Four Discounting Functions for Different Time Periods

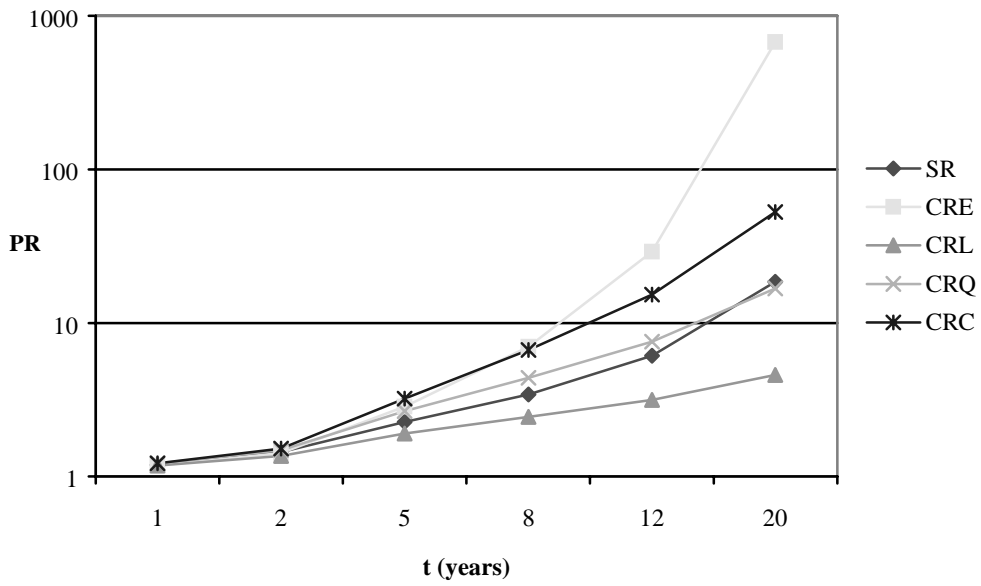


Table 3
The Price Misvaluation Coefficient (PM) by Time and Growth Rate for the Four Discounting Functions

g %	PM	TIME (YEARS)								Mean
		-5	-1	1	2	5	8	12	20	
3	PME	0.99	0.99	1.01	1.01	0.99	0.99	0.98	0.92	0.98
	PML	0.98	0.99	1.01	1.01	1.00	1.01	1.03	1.04	1.02
	PMQ	0.99	0.99	1.01	1.01	0.99	0.99	0.98	0.93	0.99
	PMC	0.99	0.99	1.01	1.01	0.99	0.99	0.98	0.91	0.98
7	PME	0.89	0.93	1.04	1.08	1.06	1.04	0.95	0.72	0.98
	PML	0.86	0.93	1.04	1.08	1.11	1.15	1.16	1.16	1.12
	PMQ	0.90	0.93	1.03	1.07	1.06	1.04	0.98	0.82	1.00
	PMC	0.90	0.93	1.03	1.07	1.05	1.03	0.93	0.73	0.97
10	PME	1.02	0.98	1.00	1.01	0.92	0.83	0.69	0.52	0.83
	PML	0.95	0.98	1.00	1.02	0.99	0.99	0.99	1.17	1.03
	PMQ	1.03	0.98	1.00	1.00	0.92	0.84	0.74	0.70	0.87
	PMC	1.04	0.98	1.00	1.00	0.90	0.81	0.68	0.56	0.82
20	PME	1.31	0.98	0.98	0.96	0.73	0.55	0.35	0.15	0.62
	PML	1.05	0.98	0.98	0.99	0.91	0.90	0.91	1.18	0.98
	PMQ	1.31	0.98	0.97	0.93	0.73	0.61	0.49	0.45	0.70
	PMC	1.40	0.98	0.97	0.93	0.68	0.52	0.36	0.25	0.62
50	PME	2.20	0.93	1.01	1.02	0.70	0.39	0.17	0.02	0.55
	PML	1.02	0.93	1.01	1.15	1.53	1.98	3.12	7.16	2.66
	PMQ	1.92	0.93	0.94	0.92	0.81	0.76	0.87	1.29	0.93
	PMC	2.68	0.93	0.92	0.86	0.58	0.42	0.36	0.35	0.58
MEAN	PME	1.28	0.96	1.01	1.02	0.88	0.76	0.63	0.47	0.79
	PML	0.97	0.96	1.01	1.05	1.11	1.21	1.44	2.34	1.36
	PMQ	1.23	0.96	0.99	0.99	0.90	0.85	0.81	0.84	0.90
	PMC	1.40	0.96	0.99	0.97	0.84	0.75	0.66	0.56	0.80

The price misvaluation (PM) coefficient is given by Eq. (11) as the ratio of SP/CP, where SP is the subjective price and CP is the computed price, CP is given by the exponential (PME), linear (PML), quadratic (PMQ) and cubic (PMC) functions. The mean column is across the forward years only.

Exhibit 3-A
The Price Misvaluation (PM) Coefficient by Growth Rates for the Four Discounting Functions

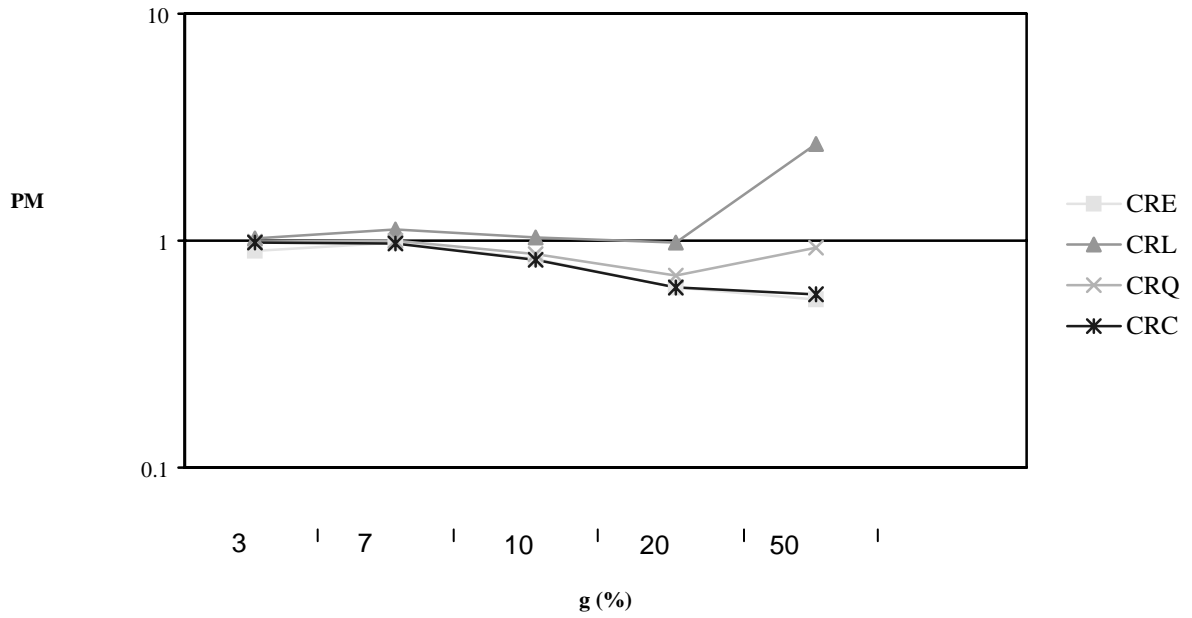


Exhibit 3-B
The Price Misvaluation (PM) Coefficient by Time for the Four Discounting Functions

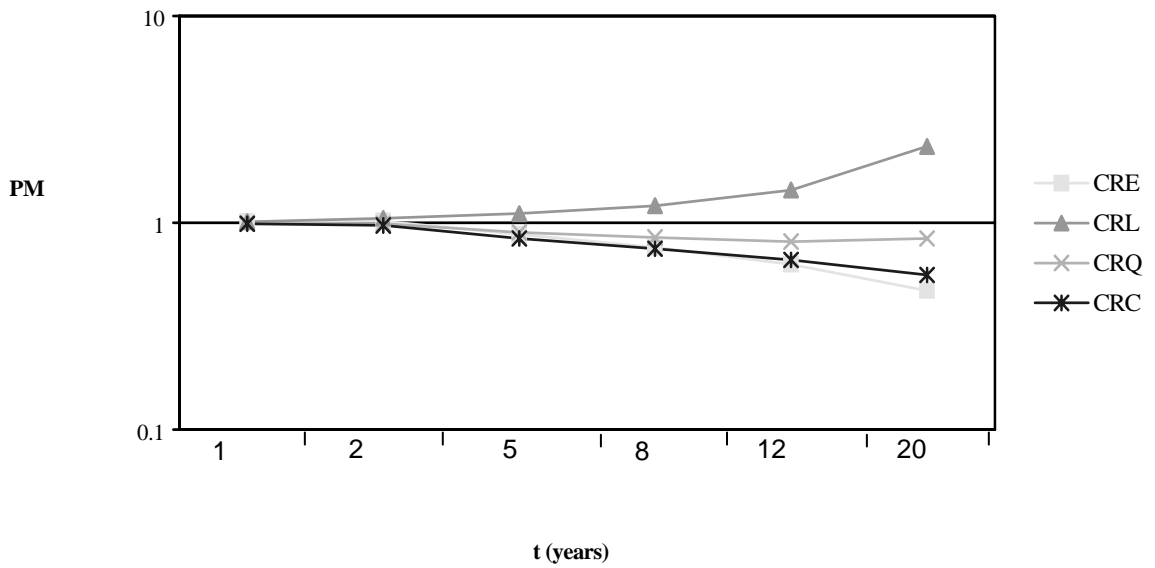


Table 4**The Subjective Growth (SG) by Time and Given Growth Rate**

g (%)	Time (years)								
	-5	-1	1	2	5	8	12	20	Mean
3	3.23	4.03	3.73	3.42	2.87	2.88	2.86	2.57	3.20
7	9.51	14.90	10.98	11.00	8.35	7.56	6.56	5.25	9.26
10	9.64	12.17	10.46	10.451	8.26	7.50	6.69	6.50	8.97
20	13.72	23.00	18.02	7.52	12.64	11.28	9.84	9.28	14.41
50	28.09	60.97	52.15	51.68	39.85	33.20	29.31	24.40	39.396
MEAN	12.84	23.01	19.07	18.82	14.39	12.48	11.05	9.60	15.16

The subjective growth rate (SG) is given by $(SP/GP)^{1/t}-1$, where GP is the given present price and t is the number of years. For the past prices, the formula is $(GP/SP)^{1/t}-1$. SG is in percent.

Table 5

The Price Ratio Dispersion (PRD) by the Exponential, Linear, Quadratic and Cubic Functions - Mean and Standard Deviation across Subjects by Time and Growth Rate

Time (Years)	PRDE			PRDL			PRDQ			PRDC		
	MEAN	SDV	COV	MEAN	SDV	COV	MEAN	SDV	COV	MEAN	SDV	COV
-5	4.08	2.09	0.51	3.71	2.08	0.56	4.08	2.09	0.51	4.26	2.11	0.49
-1	1.54	1.36	0.89	1.54	1.36	0.89	1.54	1.36	0.89	1.54	1.36	0.89
1	1.16	1.30	1.12	1.16	1.30	1.12	1.43	.19	0.83	1.47	1.17	0.79
2	2.62	2.33	0.89	2.86	2.23	0.78	3.13	2.07	0.66	3.53	1.89	0.54
5	21.38	24.74	1.16	13.43	27.94	2.08	19.39	25.35	1.31	27.06	23.48	0.87
8	78.34	30.08	0.38	29.96	42.86	1.43	43.75	32.23	0.74	73.91	29.52	0.40
12	450.22	101.56	0.23	75.89	133.71	1.76	108.93	101.89	0.94	213.39	70.76	0.33
20	12994.56	807.04	0.06	318.82	801.16	2.51	419.86	712.65	1.70	928.04	472.54	0.51
g(%)												
3	5.23	3.87	0.74	4.79	4.17	0.87	5.11	3.92	0.77	5.28	3.86	0.73
7	6.53	3.44	0.53	4.66	4.51	0.97	5.47	3.59	0.66	6.45	3.46	0.54
10	26.73	18.10	0.68	11.31	21.97	1.94	18.13	19.56	1.08	24.86	18.42	0.74
20	296.76	49.21	0.17	40.73	73.73	1.81	95.87	51.65	0.54	180.10	38.38	0.21
50	8121.24	504.90	0.06	189.33	507.47	2.68	242.41	454.01	1.87	567.02	299.87	0.53
Mean	1693.11	119.23	0.57	53.71	124.96	1.49	74.55	108.58	0.96	156.69	74.37	0.58

$$PRD_i = [(1/n) \sum_t \sum_g (SR_{i,t,g} - CR_{t,g})^2]^{1/2} \text{ where } i=1,2,\dots, I \text{ subjects; } t=1,2,\dots,T \text{ period;}$$

$g=1,2,\dots,G$ growth rates; $n=TG$. PRDE, PRDL, PRDQ and PRDC are obtained when CR is replaced with CRE, CRL, CRQ and CRC, respectively, for the exponential, linear, quadratic and cubic functions. The means, standard deviation (SDV) and coefficient of variation (COV) in the table are across the 163 subjects; where $COV = \text{Mean}/SDV$.

In the upper part of the table: $PRD_{i,t} = [(1/G) \sum_g (SR_{i,t,g} - CR_g)^2]^{1/2}$ while in the

lower part of the table: $PRD_{i,g} = [(1/T) \sum_t (SR_{i,g,t} - CR_t)^2]^{1/2}$. The Mean row in the table is given by the first equation written above.

Table 6**Regression Result of the Exponential and Linear Functions by Growth Rates****The Exponential Function**

g%	3	7	10	20	50
a	0.22	0.65	0.78	1.50	7.37
ta	6.13	13.62	8.98	9.74	2.49
b	0.81	0.58	0.42	0.11	0.02
t(b≠0)	29.89	25.97	15.86	12.44	9.90
t(b≠1)	-7.01	-18.80	-22.20	-92.74	-448.00
R ²	0.48	0.41	0.20	0.14	0.09
F	893	675	251	155	98

The Linear Function

g%	3	7	10	20	50
a	0.99	1.05	0.89	0.69	-11.3
Ta	87.9	31.53	10.97	3.49	-2.65
b	1.10	1.26	1.24	1.20	7.80
t(b≠0)	30.03	27.13	15.75	12.48	9.44
t(b≠1)	2.78	5.51	3.05	2.11	8.22
R ²	0.48	0.43	0.20	0.14	0.08
F	902	734	248	156	89

The estimated equations are (3) $(F/P) = a + b(1+g)^t$ for the exponential function, and (4) $(F/P) = a + bgt$ for the linear function, where F and P are future and present values, respectively; g is the growth rate; and t is the number of time periods. The number of observations (n) is 978 given by 163 subjects times 6 periods.

Table 7**Regression Results of the Implicit Quadratic and Cubic Functions
by Growth Rates****The Quadratic Function**

g%	3	7	10	20	50
a	0.18	0.45	0.46	0.76	-3.87
Ta	5.01	8.58	4.53	4.00	-1.08
b	0.83	0.71	0.61	0.39	1.30
t(b≠0)	29.96	26.58	15.95	12.78	10.05
t(b≠1)	-5.8	-10.65	-10.28	-19.82	2.33
R ²	0.48	0.42	0.21	0.14	0.09
F	898	707	254	163	101

The Cubic Function

g%	3	7	10	20	50
a	0.22	0.62	0.7	-1.19	1.53
ta	6.44	12.96	7.8	7.11	0.47
b	0.80	0.60	0.45	0.20	0.33
t(b≠0)	29.9	26.16	15.91	12.71	10.16
t(b≠1)	-7.52	-18.17	-19.4	-50.05	-19.8
R ²	0.42	0.41	0.20	0.14	0.09
F	707	685	253	162	103

The estimated equation is (8) $SR_{t,i} = a + b(CR_t)$; $t = 1, 2, \dots, T$ periods; $i = 1, 2, \dots, I$ subjects; for the quadratic function $CR = CRQ$, and for the cubic function $CR = CRC$. The number of observations (n) is 978 given by 163 subjects times 6 periods.

Table 8
Regression Results of the Explicit Quadratic and Cubic Functions
by Growth Rates

The Quadratic Function

g%	3	7	10	20	50
a	1.00	1.01	1.07	1.17	3.95
ta	62.02	21.37	9.37	4.17	0.65
b	1.05	1.45	0.58	0.36	-2.8
t(b≠0)	7.63	8.32	1.98	1.00	-0.90
t(b≠1)	0.39	2.58	-1.40	-1.76	-1.22
C	0.15	-0.26	0.62	0.40	2.04
t(c≠0)	0.35	-1.15	2.29	2.41	3.55
t(c≠1)	-1.97	-5.47	-1.35	-3.53	1.81
R ²	0.48	0.43	0.21	0.14	0.09
F	451	369	127	82	51

The Cubic Function

g%	3	7	10	20	50
a	1.01	1.02	0.99	0.995	0.61
ta	42.61	14.52	5.83	2.39	0.07
b	0.68	1.40	1.18	0.997	1.84
t(b≠0)	1.6	2.57	0.19	0.88	0.20
t(b≠1)	-0.72	0.73	1.05	0.00	0.08
C	3.33	-0.77	-0.93	-0.41	-0.39
t(c≠0)	0.92	-0.04	-0.40	-0.30	-0.07
t(c≠1)	0.65	-0.55	-0.55	-1.00	-0.28
d	-10.32	-0.26	-0.26	0.40	0.48
t(d≠0)	-0.89	-0.10	-0.10	0.6	0.51
t(d≠1)	-0.98	-0.47	-0.47	-0.87	-0.56
R ²	0.48	0.43	0.43	0.14	0.09
F	301	246	246	55	34

The estimated equations are (9) $SR_{t,i} = a + btg_t + C[t(t-1)/2]g_t^2 + e_{t,i}$ for the quadratic function, and (10) $SR_{t,i} = a + b + g_t + c[t(t-1)/2]g_t^2 + d[t(t-1)(t-2)/6]g_t^3 + e_{t,i}$ for the cubic function; $i=1,2,\dots,l$ subjects; $t=1,2,\dots,T$ periods.

The number of observations (n) is 978 given by 163 subjects times 6 periods.