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DERIVATION OF THE MEAN-GINI EFFICIENT PORTFOLIO FRONTIER*

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Abstract

One main advantage of the mean-variance (*MV*) portfolio frontier is its simplicity and ease of derivation. Its major shortcoming lies in its familiar restrictions, such as the quadraticity of preferences or the normality of distributions. We analytically derive the mean-Gini (*MG*) efficient portfolio frontier as a workable alternative to *MV*. If asset distributions are restricted, the *MG* frontier derivation is identical in structure to the *MV*-efficient frontier derivation. The price paid for this simplicity is that some information about the distribution of assets gets lost. We numerically derive *MG* and mean-extended Gini (*MEG*) efficient frontiers and compare the results to the *MV* frontier. *MEG* allows for the explicit introduction of risk aversion in building the efficient frontier. For U.S. classes of assets, *MG* and *MEG* efficient portfolios constructed using Ibbotson monthly returns appear to be more diversified than *MV* portfolios. When short sales are allowed, distinct investor risk aversions lead to different patterns of portfolio diversification, a result that is less obvious when short sales are foreclosed.

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1. Introduction

Since its development by Markowitz (1952, 1970), the mean-variance (*MV*) model for portfolio selection has become the standard tool by which risky financial assets are allocated. *MV* has gained a prominent place in finance because of its conceptual simplicity and ease of computation. Many authors, however, have challenged the model's premises, primarily the normality of asset return probability distributions or the quadraticity of preferences. *MV* validity has been reasserted by Levy and Markowitz (1979) and Kroll, Levy, and Markowitz (1984), who show that *MV* faithfully approximates expected utility.

The challenge to the validity of *MV* has led researchers to seek alternative solutions to efficient portfolio selection, resulting in approaches such as the three-moments, lower partial moments, semivariance, value-at-risk, stochastic dominance, and mean-Gini models, to name only a few. No other model has managed so far to attain the popularity of *MV* with practitioners, owing to the complex computation the alternative models involve.

Our purpose in this paper is twofold. First, we analytically derive the mean-Gini (*MG)* efficient frontier so it can be used as easily as *MV*. Deriving *MG* portfolios is complex mainly because of the additional information the Gini statistic infers on properties of the distributions. If one is ready to forgo this additional information, finding *MG* portfolios can be as simple as constructing *MV* portfolios. Second, we numerically compute the *MG* and the mean-extended Gini (*MEG*) efficient frontiers and compare the results to the *MV* frontier.

The *MG* approach in finance has proven to be a powerful alternative to *MV* modeling. By supplying necessary conditions for stochastic dominance, *MG* is shown to be compatible with expected utility maximization. Hence, *MG* analysis provides a consistent alternative to *MV* modeling whenever investment returns are not normally distributed or when the investor utility is not quadratic.

We developed the *MG* model of portfolio analysis in Shalit and Yitzhaki (1984), and also derived the *MG* equilibrium pricing of risky assets. No attempt, however, has been made to analytically derive the *MG*-efficient frontier, because the Gini is generally more complex to calculate in a portfolio framework.

There has been some interest in providing alternatives to the *MV*-efficient frontier that are tailored to specific investor needs. One idea is to to include risk aversion considerations in building an individual portfolio. This approach is the basis for the Black-Litterman model (1992), which uses a complex Bayesian process to include investor views in updating the prior distribution of returns.

The mean-extended Gini method offers a simple way to include risk aversion in construction of an efficient portfolio by providing an infinite number of variability measures that depend on one parameter. By changing the *MEG* parameter, the investigator modifies the risk aversion and offers an efficient frontier that suits the risk preference to the investor. The efficient frontier can then supply the price attached to a unit of risk, given a level of risk aversion.

We use the monthly returns of six United States asset classes over the last 75 years and compute the *MG*, *MEG*, and *MV* portfolio frontiers. We show that *MV*-efficient portfolios tend to be less diversified than *MG* and *MEG* portfolios. Furthermore the results for *MEG* portfolios show that, for a given required return, the proportion of stocks in the efficient portfolios declines with the degree of risk aversion as exhibited by the extended Gini parameter.

It is surprising that relying on the variance to construct efficient portfolios entails a lower level of risk aversion than would be implied using the Gini. The intuitive explanation for this result is that the portfolio distribution tends to have fat upper and lower tails. Since the variance is sensitive to both extremes, it is affected by the upper and lower tails. The effect of the upper tail is to mitigate the effect of the lower tail, leading to the unexpected result that the variance demonstrates low risk aversion (Shalit and Yitzhaki, 2002).

We first describe the *MG* and *MEG* efficient frontiers. In Section 3 we show the analytical derivation of the *MG* portfolios and compare the results to the *MV* solution. Section 4 presents the empirical application using U.S. data. Section 5 concludes the paper.

2. The Mean-Gini Efficient Frontier

In the *MG* model, investors use the portfolio's Gini as the measure of risk to be minimized, subject to a given mean return. The Gini is Gini's mean difference (*GMD*), which is defined as one-half of the expected absolute difference between the returns of two randomly drawn amounts invested in a portfolio. The Gini can also be defined as the covariance between the return and its probability distribution:

$$\Gamma = 2\mathrm{cov}[r, \ F(r)] \tag{1}$$

where r is the return, Γ the Gini, and $F(r)$ the cumulative distribution function (*CDF*).

The advantage of the Gini over the variance as a measure of risk is rooted in the necessary and sufficient conditions for second-degree stochastic dominance (*SSD*) (Yitzhaki, 1982a). Consider two portfolios (1 and 2) yielding returns r_1 and r_2, means μ_1 and μ_2, and Ginis Γ_1 and Γ_2. Then $\mu_1 \geq \mu_2$ and $\mu_1 - \Gamma_1 \geq \mu_2 - \Gamma_2$ are necessary conditions causing no risk-averse expected utility maximizer to prefer portfolio 2 to portfolio 1. If one restricts the distributions of the portfolios to the family of cumulative distributions that intersect at most once, these conditions are also sufficient. Therefore, *MG* ranks consistently risky alternatives whenever *MV* might fail (Shalit and Yitzhaki, 1984).

The implication of this result is that the efficient set of *MG* is included in the efficient set of expected utility risk-averse investors, so that every efficient *MG* portfolio maximizes the expected value of a utility function. This result does not hold for the *MV*-efficient set.

To see this, consider the choice between two portfolios. The first offers a return between zero and one dollar, and the second offers returns between one million and two million dollars. Both portfolios are included in the efficient *MV* set since the first offers a lower variance and the second a higher expected return. Thus, if one relies only on the mean and the variance, one may end up choosing the portfolio that every risk-averse expected utility investor would reject. The necessary conditions for stochastic dominance prevent *MG* users from making this mistake.

Consider a portfolio p whose returns r_p are obtained by $r_p = \boldsymbol{r}'\boldsymbol{w}$ where $\boldsymbol{r}$ is a vector of asset returns, and $\boldsymbol{w}$ is a vector of portfolio weights. Then, the Gini is written as:

$$\Gamma_p = 2\operatorname{cov}(r_p, F_p) = 2\sum w_i \operatorname{cov}(r_i, F_p) = 2\boldsymbol{w}'\boldsymbol{K}(\boldsymbol{r}, F_p) \tag{2}$$

where $\boldsymbol{K}(\boldsymbol{r}, F_p)$ is a vector of covariances of assets returns with the *CDF* of the portfolio. We can now obtain the *MG*-efficient frontier by solving the following optimization problem:

$$\begin{aligned} & Min\ \ 2\boldsymbol{w}'\boldsymbol{K}(\boldsymbol{r}, F_p) \\ & s.t.\ \ \mu_p = \boldsymbol{w}'\boldsymbol{\mu} \\ & 1 = \boldsymbol{w}'\boldsymbol{l} \\ & \boldsymbol{w} \geq 0 \end{aligned} \tag{3}$$

where, $\boldsymbol{l}$ is a vector of ones and $\boldsymbol{\mu}$ is a vector of assets mean returns. Problem (3), although similar in structure to the *MV* optimization problem, is much more complicated than the *MV* problem because the cumulative distribution is not a simple function of assets distribution functions.

MG analysis can be extended to include the investor's preference toward risk. This is done by introducing the extended Gini as a measure of risk (Yitzhaki, 1984; Shalit and Yitzhaki, 1984)[1] that attaches higher weights to the lower portions of the return probability distribution. This implies that higher risk aversion attributes more weight to the lower payoff realizations than will low risk aversion.

The extended Gini is defined much like the definition in Equation (1):

$$\Gamma(\nu) = -\nu \operatorname{cov}\{r, [1 - F(r)]^{\nu-1}\} \tag{4}$$

where ν is a parameter determining the relative weight attributed to various portions of the probability distribution. The parameter ν ranges from 1 to infinity, with ν → 1 implying variability as viewed by a risk-neutral investor, ν = 2 considering the standard Gini risk

[1] For a survey of the recent use of the extended Gini in finance, see Lien and Tse (2002).

aversion, and $\nu \to \infty$ allowing for the max-min investor who wants to avoid the worst possible outcome.

The utility function that is implied by using the extended Gini can be viewed as a special case of the utility functions suggested by Yaari's (1987) dual theory of risk aversion that distinguishes the notion of declining marginal utility of income from behavior under risk. In the case that ν is a positive integer, the link of ν to risk aversion can be shown as follows: The extended Gini equals the mean return minus the expected least outcome from ν independent random draws from the return distribution:

$$\Gamma(\nu) = \mu - E[Min(r_1,...,r_\nu)] \tag{5}$$

Equation (5) can be used to develop additional necessary conditions for stochastic dominance. In particular, comparing $\mu - \Gamma(\nu)$ of a risky portfolio with the return on a safe portfolio enables one to view $\mu - \Gamma(\nu)$ as the certainty equivalent of the portfolio. With a higher ν, one assigns a higher probability of obtaining bad outcomes. Hence, the certainty equivalent of the portfolio is lowered, and the risk premium required by the investor is higher.

This interpretation of certainty equivalence relates risk aversion to the discounting of probabilities of good outcomes, which does not originate from assuming declining marginal utility of income as in expected utility theory. Rather, the higher the risk aversion, the more the investor tends to amplify the probability of occurrence of bad events and to discount the probability of occurrence of good events.

The extended Gini of a portfolio is defined similarly to the Gini definition in Equation (2):

$$\Gamma_p(\nu) = -\nu \sum_{i=1}^{n} w_i \operatorname{cov}\{r_i, [1 - F_p(r_p)]^{\nu-1}\} \tag{6}$$

so that the optimization problem becomes:

$$\begin{aligned} &Min\, \Gamma_p(\nu) \\ &s.t.\ \mu_p = \boldsymbol{w}'\boldsymbol{\mu} \\ &1 = \boldsymbol{w}'\boldsymbol{l} \\ &\boldsymbol{w} \geq 0 \end{aligned} \tag{7}$$

In applications, the empirical cumulative distribution function is used as an estimator for the cumulative distribution. It is obtained by ranking the returns of the portfolio in increasing order, and dividing the rank of each observation by the number of observations. Since a ranking procedure is invoked each time the portfolio Gini (or extended Gini) is calculated, non-linear programming techniques should be used with caution.[2] In the application we present, we use an algorithm that does not require derivatives to find the *MG* frontier while ignoring the operational and computational efficiency of solving the problem.[3]

[2] When applied to empirical data, the optimization problem is a piece-wise linear optimization problem. See Okunev (1991), and Okunev and Dillon (1988) for a linear programming solution.

[3] The algorithm belongs to the variable metric methods. It is described in Yitzhaki (1982b).

3. Analytical Derivation of the Mean-Gini Frontier

We derive the *MG*-efficient portfolios analytically in a manner similar to derivation of *MV* portfolios. We show that if one is ready to impose restrictions on the underlying asset distributions, one can derive the *MG* portfolios using the same technique used for solving the *MV* constrained minimization problem. To see the parallel between the *MG* and the *MV* derivations, only the similarities and differences of the two dispersion measures have to be explored, because all the other components are identical.[4]

The Gini and the variance derive their properties from the covariance. The variance is calculated as the covariance of the return with itself, while the Gini is the covariance of the return with its *CDF*. Although Gini's reliance on the return and the cumulative distribution complicates its use, this relationship enables the Gini to extract more information from the underlying distribution.

The main concern in the case of portfolio analysis is that the Gini is associated with two correlation coefficients, while the variance is related to one correlation coefficient (Pearson's correlation coefficient). To see this, note that for two random variables r_i and r_j, one can define two correlation coefficients:

$$\rho_{ij} = \frac{\operatorname{cov}[r_i, F_j(r_j)]}{\operatorname{cov}[r_i, F_i(r_i)]} \qquad \rho_{ji} = \frac{\operatorname{cov}[r_j, F_i(r_i)]}{\operatorname{cov}[r_j, F_j(r_j)]} \tag{8}$$

Both correlation coefficients are needed to decompose the portfolio Gini into the contributions of individual assets to the Gini.

The properties of the Gini correlations have been examined by Schechtman and Yitzhaki (1987, 1999), who show that the correlation coefficients are equal if the distributions of r_i and r_j

[4]For a survey of the properties of the Gini, see Yitzhaki (2002).

are exchangeable up to linear transformation.[5] Intuitively, exchangeability up to linear transformation means that the shapes of marginal distributions of assets are equal up to a linear transformation. A disparity in the two correlations means a different shape in the two marginal distributions (of asset returns) needed in the correlation.[6]

The correlation coefficients allow us to decompose the portfolio Gini as follows:

Proposition: Let $r_p = \sum_{i=1}^{n} w_i r_i$. Then

$$\Gamma_P^2 - \Gamma_P \sum_{i=1}^{n} w_i D_{iP} \Gamma_i = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \Gamma_i \Gamma_j (\rho_{ij} + \rho_{ji}) \tag{9}$$

where $D_{ip} = \rho_{ip} - \rho_{pi}$ (i = 1, …, n) is the difference between the two Gini correlations defined by the return of the portfolio and the return of asset i.

Proof: The proof of the proposition is given in Wodon and Yitzhaki (2002). For completeness we provide it in appendix A.

Following exchangeability up to a linear transformation between the distribution of each asset and the portfolio, $D_{ip} = 0$. Hence, exchangeability among the assets leads to:

$$\Gamma_p^2 = \sum_{i=1}^{n} w_i^2 \Gamma_i^2 + 2 \sum_{i=1}^{n} \sum_{j \neq i}^{n} w_i w_j \Gamma_i \Gamma_j \rho_{ij} \tag{10}$$

which is identical in structure to decomposition of the variance, where each variance is substituted with a Gini, and each Pearson correlation coefficient is substituted with a symmetric Gini correlation coefficient. Since the rest of the optimization problem (3) is identical to the *MV* optimization problem, one can adapt the textbook derivation of *MV* and apply it to *MG* (see, for example, Merton, 1972, and, Huang and Litzenberger, 1988, p. 63). In other words, one can adopt the *MV* solution, allowing for short sales, and substitutes for every variance the Gini and

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[5]A set of random variables is exchangeable if for every permutation of the n subscripts, the joint distributions of ($x_{j1}, \ldots, x_{jn}$) are identical (Stuart and Ord, 1994). The multivariate normal is an example of an exchangeable distribution up to a linear transformation.

for every Pearson correlation coefficient the appropriate Gini correlation coefficient to obtain the solution to the *MG* portfolio optimization.[7] Appendix B presents the full *MG* frontier derivation in matrix notation when short sales are allowed.

Ignoring sampling variability, the two solutions will be identical if the underlying distributions are multivariate-normal. If the distribution of even one asset diverges from normality, however, the solutions of the *MG* and *MV* will differ.

The derivations presented in this section can be obtained using the extended Gini and results that pertain to the construction of efficient portfolios continue to hold.[8] However, instead of trying to verify the nature of the underlying distributions of different assets, it is easier to simply compare the optimal portfolios and see whether the change in the intensity of risk aversion makes a difference in the composition of optimal portfolios.

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[6] Further research is needed to see what can be learnt from the discrepancy in the correlations about the shape of the distributions.

[7] To be precise, for each covariance in the *MV* framework we substitute a Gini correlation multiplied by the appropriate Gini.

[8] See Schechtman and Yitzhaki (2002) for a comparison of the properties of the extended Gini correlations with the properties of the Gini correlation.

4. Comparing the *MG* and the *MV* Efficient Frontiers for U.S. Assets

We construct the efficient frontier for U.S. assets using Ibbotson's aggregate data on stocks, bonds, and bills. The data consist of 908 monthly nominal returns from January 1926 through August 2001 for six classes of assets: large-company stocks (*LCS*), small-company stocks (*SCS*), long-term corporate bonds (*LCB*), long-term government bonds (*LGB*), intermediate-term government bonds (*IGB*), and U.S. Treasury bills (*TB*). The basic statistics are given in Table 1.

As can be seen, ranking the assets according to standard deviations yields the same result as rankings by the Gini. This means that differences between assets consist mainly of magnitudes attached to risk and of correlations among the assets.[9]

The optimizations are conducted using an algorithm with a variable metric method that does not require specifying derivatives and hence is easy to implement (Yitzhaki, 1982b). Its main advantage is that it constructs the estimate of the Hessian by previous changes in the gradient and therefore is not sensitive to piecewise linearity of the target function.[10]

Table 2 displays the efficient portfolios as a function of the required return when short sales are allowed. These portfolios are simpler to interpret because the non-negativity constraints are not applied.

The first panel of Table 2 shows the share of *MV*-efficient portfolios. Note that the higher the required return, the higher the share of *LCS*, *SCS*, *LCB*, and *IGB*, and the lower the share of *LGB* and *TB*. That is, the higher the required portfolio return, the greater are the short sales of *LGB* and *TB* used to increase the shares of other assets. This expected result indicates that to increase the return of the portfolio, the investor needs to borrow by short selling.

The second panel of Table 2 presents the *MG* portfolios. The same pattern of increasing the shares of *LCS*, *SCS*, *LCB*, and *IGB* is evident, but the rate of increase is mitigated. This

[9] Note that *LCB* dominates *LGB* because of its higher mean return and lower variability. In a portfolio context, however, the statistics that matter are the covariances between assets.

[10] We do not claim this is the best algorithm available for solving the problem, as we did not try to save computer time.

means that, for a given required return, the share of *LCS* tends to be lower in *MG* than in *MV* portfolios. On the other hand, the share of *SCS* is greater in *MG* than in *MV* portfolios. The share of *LCB* is lower in *MV* than in *MG* portfolios, while *IGB* tends to have a higher share in *MV*. When we look at the assets that are used for short sales, we can see that *LGB* are used less in *MG*, while *TB* are used more aggressively in *MV*.

The last three panels of Table 2 present the efficient portfolio results for *MEG* with $\nu = 4$, for *MEG* with $\nu = 6$, and for *MEG* with $\nu = 8$. The same results observed earlier continue to hold. This means that, for a given required mean return, as risk aversion increases, the holdings of *LCS* decline and the holdings of *SCS* increase. This presents the surprising result that increasing risk aversion, as exhibited by a larger ν, implies higher shares of riskier *SCS*.

Figure 1 presents the share of stocks (*LCS* and *SCS*) in the portfolios as a function of the required rate of return. As can be seen, the higher the risk aversion, the lower the share of stocks. Note that the pattern of the *MV* portfolios implies a lower risk aversion parameter than the Gini portfolios with $\nu = 2$. This is an unexpected outcome, because as variance is a quadratic function, we expect it to have a higher weight on the extreme observations.

As Shalit and Yitzhaki (2002) point out, however, the variance imposes higher weights on *both* extremes of the distribution, which contradicts the idea that risk aversion is more concerned with the lower returns. We speculate that sensitivity of the variance to higher returns of the distribution causes the portfolios to appear as if they were constructed assuming lower values of risk aversion. This result is unexpected, as we did not anticipate differential risk aversion to cause a clear pattern in efficient portfolios.

Finally we can see that all the *MEG*-efficient portfolios meet the necessary conditions for second-degree stochastic dominance (*SSD*) since as we move to higher required returns, the mean minus the extended Gini declines, implying that no *MEG* optimal portfolio is found to be *SSD* inefficient.

Table 3 presents no short sales-allowed efficient portfolios obtained under *MV*, *MG*, and *MEG* with $\nu = 4$, 6, and 8. Under *MV* (first panel), except for the highest required return, the share of large company stock (*LCS*) is greater than the share of small company stock (*SCS*). Also, long-term government bonds (*LGB*) are not included in any of the efficient portfolios. Corporate bonds (*LCB*) tend to get the lion's share of all bond holdings whenever the required rate of return is high, while intermediate-term government bonds (*IGB*) play the same role for low required return.

The second panel of Table 3 exhibits the *MG* frontier and portfolio composition for the same selected number of mean returns. In general, it is easy to verify that the *MG* portfolios tend to be more diversified than the *MV*-efficient portfolios with the same required return. For example, in three cases, there are five assets in the *MG*-efficient portfolio, while under *MV* all optimal portfolios consist of four assets at most. Moreover, *LGB* are included in three portfolios, and in no *MV* portfolios. It is interesting that under *MG*, *LCS* have a lower share than under *MV* in almost all portfolios. The reverse can be said with respect to stocks of small companies; they have a greater share under *MG* than under *MV*. *LCB* tend to have a lower share under *MG* than under *MV*.

The final panels of Table 3 present the efficient frontier and the composition of extended Gini portfolios with $\nu = 4$, $\nu = 6$, and $\nu = 8$. The tendency for more diversified Gini portfolios continues to hold. Under $\nu = 4$, we again have three portfolios with five participating assets, and under $\nu = 6$, four portfolios with five participating assets. Also, the higher the risk aversion, the lower the share, for a given return, of *LCS*, which in most cases is compensated for with an increase in the share of *SCS*. *LGB* continue to participate in some portfolios, which is not the case for *MV*.

5. Conclusions

We have shown that the mean-Gini and mean-extended Gini portfolio frontiers can be a workable alternative to the mean-variance-efficient frontier. Besides providing necessary conditions for stochastic dominance, *MEG* analysis allows incorporation of a risk aversion differential in the construction of efficient frontier portfolios. Hence, the analyst can offer clients portfolios tailored to their risk aversion preferences. When short sales are allowed, and when return distributions are restricted to be exchangeable, *MV* and *MG* portfolios are identical in the sense that they have the same structure except that the appropriate Gini and Gini correlations are substituted for the variance and Pearson correlations.

For building U.S. asset portfolios, we use Ibbotson's aggregate data on stocks, bonds, and bills. When short sales are not allowed, *MG* and *MEG* optimal portfolios tend to be more diversified than *MV* portfolios for all levels of required mean returns. More assets are included in *MG* and *MEG*-efficient frontiers, while long-term government bonds are entirely absent from *MV* portfolios.

Appendix A: Proof of Proposition

Define w_i as the share of asset *i* in the portfolio. For simplicity, we will restrict ourselves to two assets. The extension to *n* assets is immediate. We start by substituting $r_p = w_1\, r_1 + w_2\, r_2$ into the covariance formula of the Gini:

$$\Gamma_p = 2cov[w_1 r_1 + w_2 r_2,\, F(r_p)].$$

Using the properties of the covariance we can write:

$$\Gamma_p = 2cov[w_1 r_1 + w_2 r_2,\, F(r_p)] = w_1\, 2cov[r_1,\, F(r_p)] + w_2\, 2cov[r_2,\, F(r_p)] \tag{A.1}$$
$$= w_1\, \rho_{1p}\Gamma_1 + w_2\, \rho_{2p}\, \Gamma_2$$

We define the identity:

$$\rho_{ip} = \rho_{pi} + D_{ip} \qquad \text{for } I = 1, 2 \tag{A.2}$$

where D_{ip} is the difference between the two Gini correlations defined between r_p and r_i . Using (A.1) and (A.2), we obtain:

$$\Gamma_p = w_1\, (\rho_{p1} + D_{1p})\, \Gamma_1 + a_2\, (\rho_{p2} + D_{2p})\, \Gamma_2$$

Rearranging terms:

$$\Gamma_p - w_1\, D_{1p}\, \Gamma_1 - w_2 D_{2p}\, \Gamma_2 = w_1\, \rho_{p1}\, \Gamma_1 + a_2\, \rho_{p2}\, \Gamma_2$$

Using the properties of the covariance:

$$\rho_{p1} = \frac{cov[r_p, F(r_1)]}{cov[r_p, F(r_p)]} = \frac{1}{cov[r_p, F(r_p)]}\{w_1 cov[r_1, F(r_1)] + w_2 cov[r_2, F(r_1)]\} =$$
$$= \frac{w_1\Gamma_1 + w_2\Gamma_2\rho_{21}}{\Gamma_p}\ .$$

Writing ρ_{p2} in a similar manner, we get Equation (9):

$$\Gamma_p^2 - (w_1 D_{1p}\Gamma_1 + w_2 D_{2p}\Gamma_2)\Gamma_p = w_1\Gamma_1(w_1\Gamma_1 + w_2\Gamma_2\rho_{21}) + w_2\Gamma_2(w_1\rho_{12}\Gamma_1 + w_2\Gamma_2)$$
$$= w_1^2\Gamma_1^2 + w_2^2\Gamma_2^2 + w_1 w_2\Gamma_1\Gamma_2(\rho_{12} + \rho_{21}).$$

Assuming equality of the Gini correlation coefficients between r_p and r_1 sets $D_{1p} = 0$. A similar assumption for r_2 and r_p sets $D_{2p} = 0$. The assumption $\rho = \rho_{12} = \rho_{21}$ completes the proof of (10).

Appendix B: Mean-Gini Optimization with Short Sales and Exchangeability

Under the assumption of exchangeability up to a linear transformation, between each asset and the portfolio, and among every pair of assets, the Gini of the portfolio can be written as:

$$\Gamma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \Gamma_i \Gamma_j$$

where $\rho_{ii} = 1$. In matrix notation, let **R** be the matrix of Gini correlations, **Γ** be a matrix with the Gini of the assets on the diagonal and zero in all off-diagonal terms, and **w**, be the vectors of weights. Then, the equivalent of the variance-covariance matrix is **V**:

$$\mathbf{V} = \mathbf{\Gamma}\ \mathbf{R}\ \mathbf{\Gamma}$$

and the Gini of the portfolio:

$$\Gamma_p = \boldsymbol{w}'\boldsymbol{V}\boldsymbol{w}$$

The optimization problem becomes:

$$\begin{aligned} &Min\ \boldsymbol{w}'\boldsymbol{V}\boldsymbol{w} \\ &s.t.\ \ \mu_p = \boldsymbol{w}'\boldsymbol{\mu} \\ &\quad\ 1 = \boldsymbol{w}'\boldsymbol{l} \end{aligned} \tag{B.1}$$

The problem without short sales is identical to (B.1) with the additional requirement that $\boldsymbol{w} \geq \mathbf{0}$.

The matrix **V** is a matrix of constants; hence our problem is identical to the familiar problem of minimizing the variance portfolio subject to the same restrictions. The derivation of the solution can be found in Merton (1972) and, Huang and Litzenberger (1988, pp 63-66). For completeness we give the solution to the more complicated problem of maximization subject to the non-negativity constraint.

Then

$$L(w,\lambda,\gamma) = \boldsymbol{w}'\boldsymbol{V}\boldsymbol{w} + \lambda(\mu_p - \boldsymbol{w}'\boldsymbol{\mu}) + \gamma(1 - \boldsymbol{w}'\boldsymbol{l})$$

The first-order conditions are:

$$\frac{\partial L}{\partial w} = \boldsymbol{V}\boldsymbol{w} - \lambda\boldsymbol{\mu} - \gamma\boldsymbol{l} = 0$$
$$\frac{\partial L}{\partial \lambda} = \mu_p - \boldsymbol{w'}\,\boldsymbol{\mu} = 0$$
$$\frac{\partial L}{\partial \gamma} = 1 - \boldsymbol{w'}\,\boldsymbol{l} = 0$$

yielding the solution:

$$\boldsymbol{w}_p = \boldsymbol{x} + \mu_p \boldsymbol{y} \tag{B.2}$$

where:

$$\boldsymbol{x} = B(\boldsymbol{V}^{-1}\boldsymbol{l}) - A(\boldsymbol{V}^{-1}\boldsymbol{\mu})/\,D$$
$$\boldsymbol{y} = [C(\boldsymbol{V}^{-1}\boldsymbol{\mu}) - A(\boldsymbol{V}^{-1}\boldsymbol{l})]\,/\,D$$
$$A = \boldsymbol{l}'\boldsymbol{V}^{-1}\boldsymbol{\mu}\;;\; B = \boldsymbol{\mu}'\boldsymbol{V}^{-1}\boldsymbol{\mu}\;\;;\; C = \boldsymbol{l}'\boldsymbol{V}^{-1}\boldsymbol{l}\;\;;\; D = BC - A^2$$

Equation (B.2) presents the weights of an *MG* frontier portfolio for a given μ_{pp}. Hence, similarly to *MV*, one can generate the efficient frontier by constructing convex combinations of two distinct frontier portfolios.

Consider a portfolio q built from portfolios p_1 and p_2 with weights α and (1 - α) respectively. The mean return is $\mu_q = \alpha\, \mu_1 + (1-\alpha)\, \mu_2$. By construction:

$$w_q = \alpha\, w_1 + (1-\alpha)\, w_2 = x + \mu_q\, y$$

which can be shown to yield

$$\Gamma_q^2 = \frac{1}{D}(C\mu_q^2 - 2A\mu_q + B).$$

Using this equation, one can derive the minimum Gini frontier and find out the hyperbola in the mean Gini space.

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Table 1: Nominal Monthly Returns January 1926-August 2001

Assets	Mean (%)	Std Dev (%)	Gini (%)
Large-Company Stocks (LCS)	1.0107	5.6361	2.8475
Small-Company Stocks (SCS)	1.3422	8.6259	4.2114
Long-term Corporate Bonds (LCB)	0.4893	1.9563	0.9797
Long-term Government Bonds (LGB)	0.4586	2.2149	1.1398
Intermediate Government Bonds(IGB)	0.4427	1.2540	0.6244
U.S. Treasury Bills (TB)	0.3132	0.2581	0.1417

Table 2: *MV, MG,* and *MEG* Efficient Frontiers with Short Sales Allowed (in percentages)

Mean-Variance

Mean Return	Std Dev	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.26	0.41	0.29	1.89	-1.83	1.87	97.37
0.44	0.29	5.50	3.00	11.36	-27.32	61.24	46.23
0.52	1.24	8.77	4.74	17.44	-43.69	99.35	13.40
0.69	2.25	16.05	8.61	30.99	-80.15	184.27	-59.75
0.84	3.15	22.49	12.03	42.97	-112.43	259.42	-124.49
0.94	3.74	26.66	14.25	50.74	-133.32	308.08	-166.41
1.02	4.21	30.05	16.05	57.03	-150.27	347.55	-200.40
1.10	4.68	33.42	17.84	63.31	-167.18	386.92	-234.32
1.25	5.59	39.87	21.27	75.31	-199.49	462.13	-299.11

Mean-Gini (ν = 2)

Mean Return	Gini (ν=2)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.14	0.11	0.23	1.90	-1.43	1.68	97.51
0.44	0.41	4.40	3.80	12.99	-26.59	57.42	47.95
0.52	0.64	6.88	6.22	19.91	-43.21	94.06	16.14
0.69	1.15	12.10	11.47	35.16	-79.85	175.32	-54.20
0.84	1.61	16.74	16.14	49.00	-112.65	247.49	-116.74
0.94	1.92	19.95	19.33	58.51	-135.22	297.27	-159.85
1.02	2.17	22.42	21.83	65.71	-152.73	336.37	-193.60
1.10	2.41	24.86	24.33	72.84	-170.09	374.97	-226.91
1.25	2.87	29.44	28.92	86.37	-202.40	446.80	-289.13

Mean-Extended Gini (ν = 4)

Mean Return	Gini (ν=4)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.24	0.04	0.12	2.71	-1.48	3.30	95.31
0.44	0.74	2.89	4.04	14.42	-25.46	59.88	44.23
0.52	1.17	4.48	7.00	21.29	-41.74	97.74	11.29
0.69	2.14	8.06	13.39	36.63	-77.51	181.02	-61.59
0.84	2.97	11.07	18.80	49.18	-107.62	251.88	-123.30
0.94	3.53	13.10	22.45	57.65	-127.96	299.81	-165.05
1.02	3.98	14.69	25.46	64.46	-144.40	338.77	-198.98
1.10	4.44	16.36	28.41	71.38	-160.88	377.80	-233.06
1.25	5.27	19.38	33.81	84.25	-191.49	449.75	-295.71

Table 2: *MV, MG,* and *MEG* Efficient Frontiers with Short Sales Allowed (cont.)

Mean-Extended Gini (ν = 6)

Mean Return	Gini (ν=6)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.27	0.00	0.06	2.63	-1.07	2.83	95.55
0.44	0.93	2.44	4.17	14.35	-24.76	60.91	42.87
0.52	1.47	3.73	7.31	20.47	-40.39	98.65	10.23
0.69	2.67	6.65	13.94	34.10	-74.06	180.26	-60.88
0.84	3.74	9.18	19.84	46.16	-103.65	252.24	-123.77
0.94	4.47	10.90	23.87	54.27	-123.60	301.04	-166.48
1.02	5.05	12.30	27.01	60.80	-139.60	339.86	-200.39
1.10	5.61	13.71	30.07	66.95	-154.92	377.44	-233.25
1.25	6.69	16.37	35.91	79.04	-184.46	449.51	-296.38

Mean-Extended Gini (ν = 8)

Mean Return	Gini (ν=8)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.29	-0.02	0.05	2.75	-1.1	3.03	95.28
0.44	1.07	2.36	4.25	14.21	-24.49	61.38	42.29
0.52	1.69	3.61	7.44	19.67	-39.46	98.78	9.95
0.69	3.08	6.40	14.26	32.29	-72.06	179.79	-60.68
0.84	4.30	8.80	20.28	43.20	-100.31	250.50	-122.47
0.94	5.16	10.56	24.43	50.82	-119.91	299.46	-165.37
1.02	5.82	11.99	27.60	56.80	-135.29	337.40	-198.51
1.10	6.45	13.35	30.64	62.62	-150.06	373.88	-230.42
1.25	7.70	15.99	36.61	73.64	-178.94	445.94	-293.23

Table 3: *MV, MG*,and *MEG* Efficient Frontiers with No Short Sales Allowed (in percentages)

Mean-Variance

Mean Return	Std Dev	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.26	0.31	0.29				98.49
0.44	0.84	5.26	3.96			38.14	52.64
0.52	1.33	8.55	6.39			62.89	22.18
0.69	2.52	18.96	15.30	4.29		61.45	
0.84	3.76	29.29	24.55	21.69		24.47	
0.94	4.67	36.17	30.73	33.10			
1.02	5.38	41.23	37.01	21.75			
1.10	6.12	46.30	43.29	10.41			
1.25	7.57	27.83	72.17				

Mean-Gini (ν=2)

Mean Return	Gini (ν=2)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.14	0.08	0.29	0.26		2.04	97.33
0.44	0.44	4.85	5.76	0.15		34.41	58.82
0.52	0.69	7.84	7.96	0.08		54.22	29.91
0.69	1.26	16.30	17.01	10.27	3.46	52.97	
0.84	1.92	23.66	28.20	16.96	12.52	18.66	
0.94	2.32	31.02	34.00	28.96	0.65	53.72	
1.02	2.67	31.88	43.35	12.23		12.46	
1.09	2.99	35.15	49.32	9.66			
1.25	3.72	28.08	71.95				

Mean-Extended Gini (ν=4)

Mean Return	Gini (ν=4)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.24	0.01	0.12	2.13		1.45	96.23
0.44	0.81	3.03	4.78			47.26	44.94
0.54	1.35	4.67	8.82			77.35	9.16
0.69	2.37	11.60	19.72	9.32		59.36	
0.84	3.54	15.85	33.07	24.13	0.56	26.39	
0.94	4.40	12.12	46.68	17.96		23.26	
1.02	4.99	19.34	50.39	29.75	0.10	0.42	
1.10	5.68	24.77	56.77	18.37	0.10		
1.26	7.00	24.71	74.95	0.30			

Mean-Extended Gini (ν = 6)

Mean Return	Gini (ν=6)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.27		0.07	2.20		2.56	95.17
0.44	0.97	3.09	4.56	1.71		43.03	47.62
0.52	1.57	4.48	8.35	0.00		69.85	17.33
0.69	3.02	6.80	21.93	29.12		42.16	
0.84	4.51	14.30	34.09	24.58	0.05	26.99	
0.94	5.55	15.55	43.96	35.38		5.11	
1.02	6.41	26.00	47.27	12.82	2.27	11.64	
1.10	7.30	22.55	58.86	13.82		4.78	
1.25	8.87	24.39	74.35	1.27			

Mean-Extended Gini (ν = 8)

Mean Return	Gini (ν=8)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.29		0.07	3.14		2.54	94.24
0.44	1.13	2.56	5.01	3.02		43.02	46.39
0.52	1.77	3.85	8.49	1.03		69.14	17.49
0.69	3.49	5.39	22.95	29.90	0.12	41.64	
0.84	5.20	8.77	37.07	34.79	0.03	19.34	
0.94	6.41	17.54	43.13	23.58		15.75	
1.02	7.41	14.15	54.07	30.67	1.04	0.07	
1.10	8.34	16.79	61.21	21.75		0.26	
1.25	10.27	21.91	75.73	2.37			

Figure 1: Share of LCS+SCS