

All-Pay Auctions with Risk-Averse Buyers

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Abstract

We use perturbation analysis to study independent private-value all-pay auctions with weakly risk-averse buyers. We show that in all-pay auctions, risk aversion makes buyers with low valuations bid less aggressively, particularly, less aggressively than they bid in first-price auctions. On the other hand, risk aversion makes buyers with high valuations bid more aggressively, in particular, more aggressively than they bid in first-price auctions. Nevertheless, we show that from the buyer's point of view, the all-pay auction is dominated by the first-price auction.

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1 Introduction

Auction theory has dealt mostly with risk-neutral buyers since in this case there is an explicit expression for the equilibrium bidding strategies that can be used in the analysis. Dealing with risk-averse buyers in auctions, however, is a much more complex task since the explicit expressions for equilibrium strategies in auctions with risk averse buyers cannot be obtained except for very simple models.

Several studies on the classical auction mechanisms (first-price and second-price auctions) with risk-averse buyers have appeared in the literature on auctions. In independent private-value second-price auctions, risk aversion has no effect on a buyer's optimal strategy which is to bid her own valuation for the object. In independent private-value first-price auctions, on the other hand, risk aversion makes buyers bid more aggressively (see Maskin and Riley (1984)). Thus, since the (risk-neutral) seller is indifferent to the first-price and second-price auctions when buyers are risk neutral,¹ she prefers the first-price auction to the second-price auction when buyers are risk averse. However, the seller's preference relations for auction mechanisms with risk-averse buyers do not imply anything about the buyers' preference relations for these auctions, since under risk aversion the combined revenue of the seller and the buyers is not a constant. Indeed, Matthews (1987) showed that risk averse buyers with constant absolute risk aversion are indifferent to first and second-price auctions, and that buyers prefer the first-price auction if they have increasing absolute risk aversion and the second price auction if they have decreasing

¹This result is derived from the Revenue Equivalence Theorem (Vickrey (1961), Myerson (1981), and Riley and Samuelson (1981)).

absolute risk aversion.²

In contrast to the classical auction mechanisms, relatively little is known about risk-averse buyers in all-pay auctions. Thus, the purpose of this paper is to study independent private-value all-pay auctions from the point of view of risk-averse buyers. In the all-pay auction as in the first-price auction, the highest bidder wins, but while in the first-price auction only the highest bidder pays her bid, in the all-pay auction all the buyers pay their bids.³ We consider a situation where initially all buyers are risk neutral (henceforth referred to as the status quo), but afterwards the utility function of each buyer undergoes a mild independent change such that her new utility function is concave, that is, buyers become weakly risk averse. Then, we employ perturbation analysis, one of the most powerful tools in applied mathematics, to calculate an approximation of the equilibrium strategies of risk-averse buyers in auctions. As we shall see, such approximate solutions can be very insightful for auctions with risk-averse buyers, making the sacrifice of ‘exactness’ worthwhile.

The paper is organized as follows. In Section 2 we calculate the equilibrium strategies of risk-averse buyers in first-price auctions. From the expressions of the equilibrium strategies we can immediately recover the well-known effect of risk aversion on the equilibrium bids, namely, risk-averse buyers bid more aggressively than they bid in the status quo.

In Section 3 we show that the effect of risk aversion on the bids in all-pay auctions is

²This result was generalized first by Monderer and Tennenhltz (2000) to all k-price auctions and later by Hon-Snir (2001) to the auction mechanisms for which the Revenue Equivalence Theorem holds.

³Several applications of all-pay auctions have been made to job-promotions competitions, R&D competitions, political campaigns, political lobbying, sport competitions, etc.

quite complex. On one hand, a risk-averse buyer with a low valuation bids less aggressively than she bids in the status quo. On the other hand, a risk-averse buyer with a high valuation bids more aggressively than she bids in the status quo.

In Section 4 we compare all-pay auctions with first-price auctions from the risk-averse buyers' point of view. Intuitively, one can expect that as in the risk-neutral case, the equilibrium bids of risk averse buyers in all-pay auctions should be lower than in first-price auctions. We show that indeed in all-pay auctions low types bid less aggressively than they bid in first-price auctions. Surprisingly, however, high types bid more aggressively in all-pay auctions than they bid in first-price auctions. Thus, it is not clear in which auction the (ex-ante) buyer's expected utility is larger. Nevertheless, we show that, independent of the distribution of the buyers' valuations and the number of buyers, the expected payoff of every buyer in the first-price auction is always larger than her expected payoff in the all-pay auction. Consequently, a risk-averse buyer will prefer the first-price auction to the all-pay auction. The dominance of the first-price auction from the buyers' point of view can be generalized to any auction mechanism in which the buyer pays part of her bid whether or not she wins and she pays the rest of the bid only if she wins. However, the seller's preference relation among first-price and all-pay auctions under risk aversion remains an open question.⁴

⁴Maskin and Riley (1984) showed that the seller's expected payoff in the first-price auction is larger than her expected payoff in the all-pay auction for some very special utility functions.

2 First-price auctions

Consider n buyers that compete to acquire a single object in a first-price auction. The valuation of each buyer for the object v is independently distributed according to a distribution function $F(v)$ on the interval $[\underline{v}, \bar{v}]$. Each buyer places a bid b and the highest buyer wins the object and pays her bid. Each buyer's utility is given by the function $U(v - b)$, which is twice continuously differentiable, monotonically increasing, normalized such that $U(0) = 0$, and satisfies $U'' \leq 0$ (i.e., risk-averse or risk-neutral buyers).

Since the equilibrium bid function $b(v)$ is monotonically increasing (Maskin and Riley (2000)), we can define the equilibrium inverse bid function as $v = v(b)$. The maximization problem of buyer i with valuation v is given by

$$\max_b V_i = F^{n-1}(v(b))U(v - b) .$$

Differentiating with respect to b gives

$$\frac{\partial V_i}{\partial b} = (n - 1)F^{n-2}(v(b))f(v(b))v'(b)U(v - b) - U'(v - b)F^{n-1}(v(b)) = 0.$$

Therefore,

$$v'(b) = \frac{1}{n - 1} \frac{F(v(b))}{f(v(b))} \frac{U'(v(b) - b)}{U(v(b) - b)} , \tag{1}$$

where $f = F'$ is the density. Since the lowest type \underline{v} has zero utility, the initial condition for equation (1) is given by

$$v(b = \underline{v}) = \underline{v}. \tag{2}$$

Equation (1) is exact in the risk-neutral case, i.e., $U(x) = x$. In that case this equation can be solved explicitly as follows

$$b_{rn}^{1st}(v) = v - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) ds . \quad (3)$$

There are no such explicit solutions for a general utility function U . Hence, we consider the case of weak risk aversion, i.e., when U is given by

$$U(x) = x + \varepsilon u(x) , \quad \varepsilon \ll 1 . \quad (4)$$

Thus, ε is the risk aversion parameter and $\varepsilon \ll 1$ implies weak risk aversion. Note that $u(0) = 0$ and $u'' \leq 0$.

Under the assumption of weak risk aversion we can use perturbation analysis to obtain explicit expressions of the equilibrium bid functions in first-price auctions as follows.

Proposition 1 *The symmetric equilibrium bid function in a first-price auction with risk-averse buyers is given by*

$$b^{1st}(v) = b_{rn}^{1st}(v) + \varepsilon b_1^{1st}(v) + O(\varepsilon^2), \quad (5)$$

where $b_{rn}^{1st}(v)$ is the risk-neutral equilibrium strategy (3),

$$b_1^{1st}(v) = \frac{-1}{F^{n-1}(v)} \int_{\underline{v}}^{b_{rn}^{1st}(v)} F^{n-1}(v_{rn}^{1st}(b)) \left[u'(v_{rn}^{1st}(b) - b) - \frac{u(v_{rn}^{1st}(b) - b)}{v_{rn}^{1st}(b) - b} \right] db , \quad (6)$$

and $v_{rn}^{1st}(b)$ is the inverse function of (3).

Proof: See Appendix A.

The expression for $b_1^{1st}(v)$ can be rewritten as follows.

Corollary 1

$$b_1^{1st}(v) = u(v - b_{rn}^{1st}(v)) - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}^{1st}(s)) ds. \quad (7)$$

Proof. See Appendix B.

The “payoff” for the lengthy calculations which are needed to derive the explicit expression of the equilibrium bids in Proposition 1 is that it enables us to analyze the role of weak risk aversion in first-price auctions and to derive some conclusions which, when derived directly from the differential equation model (1), requires considerably more work.

The first consequence that we can obtain immediately from (6) is⁵

Corollary 2 *In a first-price auction the equilibrium bid of every risk-averse buyer with type v is larger than the equilibrium bid of a risk-neutral buyer with type v .*

Proof. By the mean value theorem

$$u'(v_{rn}^{1st}(b) - b) - \frac{u(v_{rn}^{1st}(b) - b)}{v_{rn}^{1st}(b) - b} = u''(\zeta_1(b))\zeta_2(b), \quad 0 \leq \zeta_1(b) \leq \zeta_2(b) \leq v_{rn}^{1st}(b) - b.$$

Thus, we can rewrite (6) as

$$b_1^{1st}(v) = \frac{-1}{F^{n-1}(v)} \int_{\underline{v}}^{b_{rn}^{1st}(v)} F^{n-1}(v_{rn}^{1st}(b)) [u''(\zeta_1(b))\zeta_2(b)] db.$$

Since $u'' \leq 0$, it follows that $b_1^{1st}(v) \geq 0$ for all v . \square

An immediate consequence of Corollary 2 is that the seller’s expected revenue is higher in the case of risk aversion than in the risk-neutral case.

⁵This result has been established by Maskin and Riley (1984).

We now use the explicit expression obtained in Proposition 1 to analyze the effect of weak risk aversion on the buyers' ex-ante utility.

Proposition 2 *The risk-averse buyer's (ex-ante) payoff in a first-price auction is given by*

$$V^{1st}(\varepsilon) = V_{rn} + \varepsilon \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(1 - F(v))u'(v - b_{rn}^{1st}(v)) dv + O(\varepsilon^2) ,$$

where $V_{rn} = \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(v - b_{rn}^{1st}(v)) f(v) dv$ is the (ex-ante) expected payoff in the risk-neutral case.

Proof: See Appendix C. \square

By Proposition 2, we can observe the following sufficient conditions for the preference relations among the buyers' expected payoffs in the risk-neutral and risk-averse cases.

Corollary 3 *In a first-price auction,*

- *If $u'(x) > 0$ for all x , then the risk-averse buyer's (ex-ante) expected payoff is larger than the (ex-ante) expected payoff of a risk-neutral buyer, i.e., $V^{1st}(\varepsilon) > V_{rn}$.*
- *If $u'(x) < 0$ for all x , then the risk-averse buyer's (ex-ante) expected payoff is smaller than the (ex-ante) expected payoff of a risk-neutral buyer, i.e., $V^{1st}(\varepsilon) < V_{rn}$.*

It is interesting to note that if $u > 0$, the expected utility of every risk-averse type is not necessarily larger than her expected utility in the risk-neutral case. To see that, let us consider the case where $n = 2$, $F(v) = v$, $v \in [0, 1]$, $u(x) = x^\beta(1 - x)$, and $0 < \beta < 1$.

Then, $b_{rn}(v) = v/2$, $u > 0$ and $u'' < 0$. From Proposition 2 we have that

$$\begin{aligned} V^{1st}(\varepsilon) - V_{rn} &\approx \varepsilon \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(1 - F(v))u'(v - b_{rn}(v))dv = \\ &= \varepsilon \int_0^1 v(1 - v) \left(\beta \left(\frac{v}{2}\right)^{\beta-1} - (\beta + 1) \left(\frac{v}{2}\right)^\beta \right) dv = \\ &= \frac{-\varepsilon(1 - \beta^2 - 4\beta)}{2^\beta(\beta + 1)(\beta + 2)(\beta + 3)}. \end{aligned}$$

We can see that although $u > 0$, $V^{1st}(\varepsilon) < V_{rn}$ when $0 < \beta < \sqrt{5} - 2$.

3 All-pay auctions

Consider n buyers that compete to acquire a single object in an all-pay auction. The valuation of each buyer for the object v is independently distributed according to a distribution function $F(v)$ on the interval $[\underline{v}, \bar{v}]$. Each buyer submits a bid b and pays her bid regardless of whether she wins or not, but only the highest buyer wins the object. The maximization problem of buyer i with valuation v is given by

$$\max_b V_i = F^{n-1}(v(b))U(v - b) + (1 - F^{n-1}(v(b)))U(-b).$$

Differentiating with respect to b gives

$$\begin{aligned} \frac{\partial V_i}{\partial b} &= (n - 1)F^{n-2}(v(b))f(v(b))v'(b)[U(v - b) - U(-b)] \\ &\quad - F^{n-1}(v(b))[U'(v - b) - U'(-b)] - U'(-b) = 0. \end{aligned}$$

Therefore, the inverse bid function satisfies the ODE

$$\begin{aligned} v'(b) &= \frac{F(v(b))[U'(v - b) - U'(-b)]}{(n - 1)f(v(b))[U(v - b) - U(-b)]} \\ &\quad + \frac{U'(-b)}{(n - 1)F^{n-2}(v(b))f(v(b))[U(v - b) - U(-b)]}, \end{aligned} \tag{8}$$

with the initial condition $v(0) = \underline{v}$. As with first-price auctions, equation (8) is exact in the risk-neutral case $U(x) = x$. In that case it can be solved explicitly and yields

$$b_{rn}^{all}(v) = vF^{n-1}(v) - \int_{\underline{v}}^v F^{n-1}(s) ds. \quad (9)$$

There are no such explicit solutions for a general utility function U . As before, however, we can use perturbation analysis to obtain an explicit solution for the case of weak risk aversion.

Proposition 3 *The symmetric equilibrium bid function in an all-pay auction with risk-averse buyers is given by*

$$b^{all}(v) = b_{rn}^{all}(v) + \varepsilon b_1^{all}(v) + O(\varepsilon^2),$$

where $b_{rn}^{all}(v)$ is the equilibrium bid in the risk-neutral case (9), and

$$\begin{aligned} b_1^{all}(v) = & u(-b_{rn}^{all}(v)) + F^{n-1}(v) \left[u(v - b_{rn}^{all}(v)) - u(-b_{rn}^{all}(v)) \right] \\ & - \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}^{all}(s)) ds. \end{aligned} \quad (10)$$

Proof. See Appendix D. \square

We can use the expression for $b^{all}(v)$ to show that risk aversion affects low type buyers to bid less aggressively.

Corollary 4 *In an all-pay auction the equilibrium bid of a risk-averse buyer with low type v is smaller than the equilibrium bid of a risk-neutral buyer with type v .*

Proof. We prove this result by showing that $b_1^{all}(v) < 0$ if v is sufficiently close to \underline{v} .

To see that, we first note that from (9) it follows that

$$(b_{rn}^{all})'(v) = (n-1)vF^{n-2}(v)f(v). \quad (11)$$

Differentiating $b_1^{all}(v)$ in (10) and using (11) yields

$$(b_1^{all})'(v) = (n-1)F^{n-2}(v)f(v)S(v),$$

where

$$S(v) = -v \left[F^{n-1}(v)u'(v - b_{rn}^{all}) + (1 - F^{n-1}(v))u'(-b_{rn}^{all}) \right] + u(v - b_{rn}^{all}) - u(-b_{rn}^{all}).$$

In order to complete the proof it is sufficient to show that $S(v) < 0$ if v is sufficiently close to \underline{v} . Indeed, in this case,

$$S(v) \approx -vu'(-b_{rn}) + u(v - b_{rn}) - u(-b_{rn}) = -v[u'(-b_{rn}) - u'(x)],$$

where $x \in (-b_{rn}, v - b_{rn})$. Since $x > -b_{rn}$, the concavity of u implies that $u'(-b_{rn}) - u'(x) > 0$. \square

The following result shows that risk aversion affects high type buyers and low type buyers quite differently.

Corollary 5 *In an all-pay auction, the equilibrium bid of a risk-averse buyer with high type v is larger than the equilibrium bid of a risk-neutral buyer with type v .*

Proof. From Corollary 2 we have that $b_1^{1st}(\bar{v}) \geq 0$. In addition, in Proposition 6 we show that $b_1^{all}(\bar{v}) \geq b_1^{1st}(\bar{v})$. Therefore, $b_1^{all}(\bar{v}) \geq 0$. \square

We now use the explicit expression obtained in Proposition 3 to analyze the effect of weak risk aversion on the buyers' ex-ante utility.

Proposition 4 *The risk averse buyer's (ex-ante) payoff in an all-pay auction is given by*

$$V^{all} = V_{rn} + \varepsilon \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(1 - F(v))u'(v - b_{rn}^{all}(v)) dv + O(\varepsilon^2).$$

where V_{rn} is the (ex-ante) expected payoff in the risk-neutral case.

Proof: See Appendix E. \square

Proposition 4 implies that the result of Corollary 3 is also true for all-pay auctions:

Corollary 6 *In an all-pay auction,*

- *If $u'(x) > 0$ for all x , then the risk-averse buyer's (ex-ante) expected payoff is larger than the (ex-ante) expected payoff of a risk-neutral buyer, i.e., $V^{all}(\varepsilon) > V_{rn}$.*
- *If $u'(x) < 0$ for all x , then the risk-averse buyer's (ex-ante) expected payoff is smaller than the (ex-ante) expected payoff of a risk-neutral buyer, i.e., $V^{all}(\varepsilon) < V_{rn}$.*

Similarly to the case of first-price auctions, it can be shown that if $u > 0$, the expected payoff of every risk-averse buyer is not necessarily higher than her expected utility in the risk-neutral case.

4 First-price auctions versus all-pay auctions

One way to gain insight into all-pay auctions is to compare them with first-price auctions.

Since in an all-pay auction a buyer pays her bid whether or not she wins and in a first-price

auction she pays only if she wins, it seems natural to expect that buyers will be more careful (i.e., have lower bids) in all-pay auctions than in first-price auctions. Indeed, it is well known that the bid of a risk-neutral buyer in an all-pay auction is smaller than her bid in a first-price auction and we can expect this relation to be even stronger for risk-averse buyers. However, as we show below the relation of bids in first-price and all-pay auctions is not so simple.

Proposition 5 *The equilibrium bid of a risk-averse buyer with low type v in an all-pay auction is smaller than her bid in a first-price auction, that is,*

$$b^{all}(v) \leq b^{1st}(v) \quad \text{for } 0 \leq v - \underline{v} \ll 1.$$

Proof. It is well known that in the risk-neutral case for every type v , $b_{rn}^{all}(v) \leq b_{rn}^{1st}(v)$. In addition, from Corollary 3 and Corollary 4 we have that for v sufficiently close to \underline{v} , $b_1^{1st}(v) \geq 0$ and $b_1^{all}(v) \leq 0$. \square

The following result shows that risk-averse buyers in all-pay auctions bid more aggressively than in first-price auctions.

Proposition 6 *The equilibrium bid of a risk-averse buyer with high type v in an all-pay auction is larger than in a first-price auction. In particular,*

$$b^{all}(\bar{v}) > b^{1st}(\bar{v}).$$

Proof: Since $b_{rn}^{all}(\bar{v}) = b_{rn}^{1st}(\bar{v})$, it is sufficient to show that $b_1^{all}(\bar{v}) \geq b_1^{1st}(\bar{v})$. From (7) and (10),

$$b_1^{all}(\bar{v}) - b_1^{1st}(\bar{v}) = - \int_{\underline{v}}^{\bar{v}} F^{n-1}(v) \left[u'(v - b_{rn}^{all}(v)) - u'(v - b_{rn}^{1st}(v)) \right] dv.$$

In addition, since $b_{rn}^{1st}(v) \geq b_{rn}^{all}(v)$ for all v , by the concavity of u we have that $u'(v - b_{rn}^{all}(v)) \leq u'(v - b_{rn}^{1st}(v))$. Hence, the result follows. \square

We thus conclude that there is no dominance relation among the bids in first-price and all-pay auctions. Nevertheless, first-price auctions dominate all-pay auctions from the buyer's point of view.

Proposition 7 *The (ex-ante) expected payoff of every buyer in the first-price auction is larger than her (ex-ante) expected payoff in the all-pay auction.*

Proof. By the Revenue Equivalence Theorem, the expected payoff of a risk-neutral buyer with valuation v is the same in first-price auctions and all-pay auctions. Thus, we obtain that the difference between her expected payoffs in these auctions in the case where all types are risk-averse is

$$V^{1st} - V^{all} = \epsilon \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(1 - F(v))(u'(v - b_{rn}^{1st}(v)) - u'(v - b_{rn}^{all}(v))) + O(\epsilon^2).$$

Since $v - b_{rn}^{1st}(v) \leq v - b_{rn}^{all}(v)$ for all v and since u is concave, then $u'(v - b_{rn}^{1st}(v)) \geq u'(v - b_{rn}^{all}(v))$ and therefore $V^{1st} - V^{all} > 0$. \square

5 Example

The results of the perturbation analysis are illustrated with the following example. Consider two bidders where each bidder's valuation is distributed on $[0, 1]$ according to the distribution function $F(v) = v^\alpha$. Assume that each bidder's utility function is $U(x) = x - \epsilon x^2$.

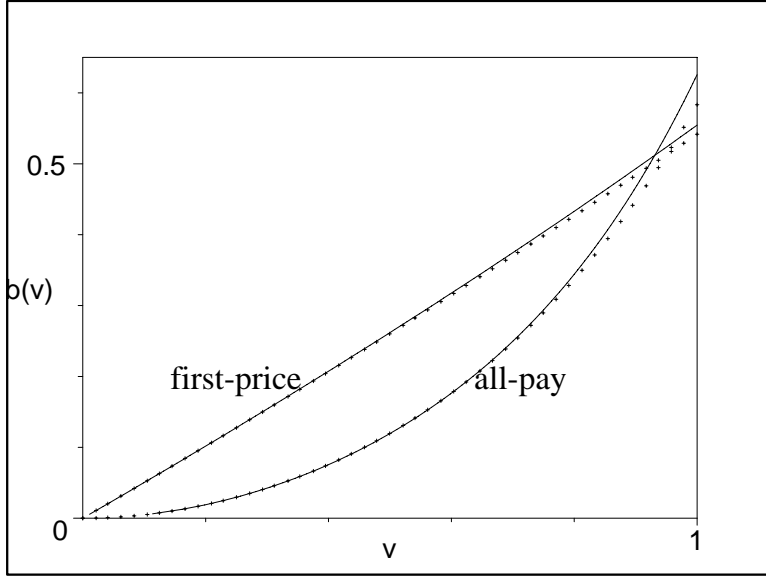


Figure 1: Bids of risk-averse buyers (solid lines) and their explicit approximations (dotted lines) in first-price and all-pay auctions.

From Proposition 1 we have that the equilibrium bid function in the first-price auction is given by

$$b^{1st}(v) = \frac{\alpha}{1+\alpha}v + \varepsilon \frac{\alpha}{(1+\alpha)^2(2+\alpha)}v^2 + O(\varepsilon^2). \quad (12)$$

Similarly, from Proposition 3 we have that the equilibrium bid function in the all-pay auction is given by

$$b^{all}(v) = \frac{\alpha}{1+\alpha}v^{1+\alpha} + \varepsilon \left(-\frac{\alpha}{2+\alpha}v^{2+\alpha} + \frac{\alpha}{1+\alpha}v^{2+2\alpha} \right) + O(\varepsilon^2). \quad (13)$$

In Figure 1 we compare the approximations (12) and (13) with the exact bid functions (i.e., the numerical solutions of equations (1) and (8), respectively), for the case $\alpha = 1$ (i.e., uniform distribution) and $\varepsilon = 0.5$. Although the risk-aversion parameter is not very small, the agreement between the explicit approximations and the exact values is quite

remarkable. Such a good agreement was also observed in numerous other comparisons that we carried out with different distribution functions and utility functions.

In Figure 2 we compare the (exact) bid functions in the risk-averse and risk-neutral case. As predicted by the perturbation analysis, under risk aversion the bids increase for all types in first-price auctions, whereas in all-pay auctions risk aversion lowers the bids of the low types but increases the bids of the high-types. In particular, under risk aversion the low types bid less aggressively in all-pay auctions than in first-price auctions, but the high types bid more aggressively in all-pay auctions than in first-price auctions.

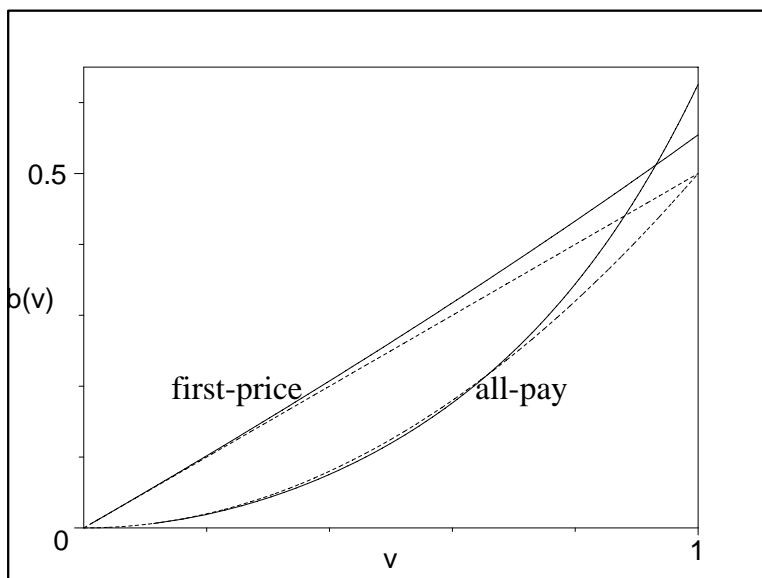


Figure 2: Bids of risk-averse buyers (solid lines) and of risk-neutral buyers (dashed lines) in first-price and all-pay auctions.

References

- [1] Hon-Snir, S. (2001): “Utility Equivalence in Auctions,” mimeo.
- [2] Maskin, E. and Riley, J. G. (1984): “Optimal Auctions with Risk Averse Buyers,” *Econometrica*, 6, 1473-1518.
- [3] Maskin, E. S. and Riley, J. G. (2000): “Equilibrium in Sealed High Bid Auctions,” *Review of Economic Studies*, 67, 439-454.
- [4] Matthews, S. (1987): “Comparing Auctions for Risk Averse Buyers: A Buyer’s Point of View,” *Econometrica*, 55, 636-646.
- [5] Monderer, D. and Tennenholtz, M. (2000): “K-Price Auctions,” *Games and Economic Behavior*, 31, 220-244.
- [6] Myerson, R. B. (1981): “Optimal Auction Design,” *Mathematics of Operations Research*, 6, 58-73.
- [7] Riley, J. G. and Samuelson, W. F. (1981): “Optimal Auctions,” *American Economic Review*, 71, 381-392.
- [8] Vickrey, W. (1961): “Counterspeculation, Auctions, and Competitive Sealed Tenders,” *Journal of Finance*, 16, 8-37

A Proof of Proposition 1

We need the following Lemma,⁶

Lemma 1 *Let $v(b)$ be the inverse bid function in a first-price auction, which satisfies the conditions of Proposition 1. Then, $v(b) = v_{rn}(b) + \varepsilon v_1(b) + O(\varepsilon^2)$, where $v_{rn}(b)$ is the inverse function of (3) and*

$$v_1(b) = \frac{v'_{rn}(b)}{F^{n-1}(v_{rn}(b))} \int_v^b F^{n-1}(v_{rn}(b)) \left[u'(v_{rn}(b) - b) - \frac{u(v_{rn}(b) - b)}{v_{rn}(b) - b} \right] db. \quad (14)$$

In addition, we note that if we differentiate with respect to ε , the identity $v = v(b(v; \varepsilon); \varepsilon)$ and set $\varepsilon = 0$, we get that

$$v_1(b_{rn}(v)) + v'_{rn}(b_{rn}(v))b_1(v) = 0. \quad (15)$$

Substitution of v_1 from (14) and v'_{rn} in (eq:b1_v1) completes the proof of Proposition 1.

□

A.1 Proof of Lemma 1

Substituting (4) in (1) yields

$$v'(b) = \frac{F(v(b))}{(n-1)f(v(b))} \frac{1 + \varepsilon u'(v(b) - b)}{(v(b) - b + \varepsilon u(v(b) - b))}. \quad (16)$$

We can write the equilibrium bid as $v(b) = v_{rn}(b) + \varepsilon v_1(b) + O(\varepsilon^2)$, where $v_{rn}(b)$ is the inverse function of the risk-neutral equilibrium strategy in first-price auctions (3). Note

⁶For clarity, we drop the superscripts *1st* and *all* from the proofs.

that, for clarity, we drop the superscript *1st*. We first note that when $\varepsilon \ll 1$,

$$\begin{aligned}
F(v(b)) &= F(v_{rn}(b)) + \varepsilon v_1(b)F'(v_{rn}(b)) + O(\varepsilon^2), \\
f(v(b)) &= f(v_{rn}(b)) + \varepsilon v_1(b)f'(v_{rn}(b)) + O(\varepsilon^2), \\
\varepsilon u(v(b) - b) &= \varepsilon u(v_{rn}(b) - b) + O(\varepsilon^2), \\
\varepsilon u'(v(b) - b) &= \varepsilon u'(v_{rn}(b) - b) + O(\varepsilon^2).
\end{aligned} \tag{17}$$

Substituting $v(b) = v_{rn}(b) + \varepsilon v_1(b) + O(\varepsilon^2)$ and (17) in (16) and expanding in a power series in ε gives,

$$\begin{aligned}
(v_{rn})'(b) + \varepsilon (v_1)'(b) &= \frac{1}{n-1} \frac{F(v_{rn}(b)) + \varepsilon v_1(b)f(v_{rn}(b))}{f(v_{rn}(b))} \left(1 - \varepsilon v_1(b) \frac{f'(v_{rn}(b))}{f(v_{rn}(b))} \right) \times \\
&\quad \frac{1 + \varepsilon u'(v_{rn}(b) - b)}{v_{rn}(b) - b} \left(1 - \varepsilon \frac{v_1(b) + u(v_{rn}(b) - b)}{v_{rn}(b) - b} \right) + O(\varepsilon^2).
\end{aligned}$$

By construction, the equation for the $O(1)$ terms is identical to the risk-neutral case and thus, automatically satisfied. The equation for the $O(\varepsilon)$ terms is

$$\begin{aligned}
(v_1)'(b) &= \frac{1}{n-1} \frac{F(v_{rn}(b))}{f(v_{rn}(b))} \left[\frac{u'(v_{rn}(b) - b)}{v_{rn}(b) - b} - \frac{v_1(b) + u(v_{rn}(b) - b)}{(v_{rn}(b) - b)^2} \right] \\
&\quad - \frac{1}{n-1} v_1(b) \frac{F(v_{rn}(b))f'(v_{rn}(b))}{f^2(v_{rn}(b))} \frac{1}{(v_{rn}(b) - b)} + \frac{1}{n-1} v_1(b) \frac{1}{(v_{rn}(b) - b)},
\end{aligned}$$

subject to the initial condition $v_1(\underline{v}) = 0$. This equation can be rewritten as

$$(v_1)'(b) + v_1(b)A(b) = D(b),$$

where

$$\begin{aligned}
A(b) &= \frac{1}{(n-1)(v_{rn}(b) - b)} \left(\frac{F(v_{rn}(b))f'(v_{rn}(b))}{f^2(v_{rn}(b))} - 1 + \frac{F(v_{rn}(b))}{f(v_{rn}(b))(v_{rn}(b) - b)} \right), \\
D(b) &= \frac{F(v_{rn}(b))}{(n-1)f(v_{rn}(b))} \left(-\frac{u(v_{rn}(b) - b)}{(v_{rn}(b) - b)^2} + \frac{u'(v_{rn}(b) - b)}{v_{rn}(b) - b} \right).
\end{aligned}$$

The solution of this equation is

$$v_1(b) = e^{\int_b^{\bar{b}_{rn}} A} \left(C - \int_b^{\bar{b}_{rn}} D(x) e^{-\int_x^{\bar{b}_{rn}} A} dx \right), \quad (18)$$

where C is a constant and $\bar{b}_{rn} = b_{rn}(\bar{v}) = \bar{v} - \int_{\underline{v}}^{\bar{v}} F^{n-1}(s) ds$ is the maximal bid in the risk-neutral case, which is obtained from (3). It can be verified that

$$e^{\int_b^{\bar{b}_{rn}} A} = \frac{f(\bar{v})F(v_{rn}(b)) \int_{\underline{v}}^{\bar{v}} F^{n-1}(s) ds}{f(v_{rn}(b)) \int_{\underline{v}}^{v_{rn}(b)} F^{n-1}(s) ds}.$$

Therefore $\lim_{b \rightarrow \underline{v}} e^{\int_b^{\bar{b}_{rn}} A} = \infty$. We thus conclude that $C = \int_{\underline{v}}^{\bar{b}_{rn}} D(x) e^{-\int_x^{\bar{b}_{rn}} A} dx$. Therefore

$$v_1(b) = \frac{F(v_{rn}(b))}{(n-1)f(v_{rn}(b)) \int_{\underline{v}}^{v_{rn}(b)} F^{n-1}(s) ds} \int_{\underline{v}}^b F^{n-1}(v_{rn}(b)) \left[u'(v_{rn}(b) - b) - \frac{u(v_{rn}(b) - b)}{v_{rn}(b) - b} \right] db.$$

The proof is completed since in the risk-neutral case we have from (1) that $v'_{rn}(b) =$

$\frac{F(v_{rn}(b))}{(n-1)f(v_{rn}(b))} \frac{1}{v_{rn}(b)-b}$, which combined with (3) gives that

$$v'_{rn}(b) = \frac{F^n(v_{rn}(b))}{(n-1)f(v_{rn}(b)) \int_{\underline{v}}^{v_{rn}(b)} F^{n-1}(s) ds}. \quad (19)$$

□

B Proof of Corollary 1

After changing variables, from (6) we have,

$$\begin{aligned} b_1(v) &= \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) \frac{u(s - b_{rn}(s))}{s - b_{rn}(s)} \frac{db_{rn}(s)}{ds} ds - \\ &\quad \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) \frac{db_{rn}(s)}{ds} ds. \end{aligned}$$

Substituting $\frac{db_{rn}(v)}{dv} = (n-1)\frac{f(v)}{F(v)}(v - b_{rn}(v))$, (see (1)) in the first integral and integrating by parts gives

$$\begin{aligned}
b_{rn} &= \frac{(n-1)}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-2}(s) f(v) u(s - b_{rn}(s)) ds - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) \frac{db_{rn}(s)}{ds} ds \\
&= u(v - b_{rn}(v)) - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) \left(1 - \frac{db_{rn}(s)}{ds}\right) ds \\
&\quad - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) \frac{db_{rn}(s)}{ds} ds \\
&= u(v - b_{rn}(v)) - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) ds.
\end{aligned}$$

C Proof of Proposition 2

By (4), $V^{1st} = \int_{\underline{v}}^{\bar{v}} F^{n-1}(v) [v - b(v) + \varepsilon u(v - b(v))] f(v) dv$. If we substitute $b(v) = b_{rn}(v) + \varepsilon b_1(v) + O(\varepsilon^2)$ and expand in a power series in ε we get that

$$V^{1st} = V_{rn}^{1st} + \varepsilon G + O(\varepsilon^2), \quad G = \int_{\underline{v}}^{\bar{v}} F^{n-1}(v) [-b_1(v) + u(v - b_{rn}(v))] f(v) dv.$$

Substituting $b_1(v)$ from (6) yields

$$\begin{aligned}
G &= \int_{\underline{v}}^{\bar{v}} \left\{ \left(\int_{\underline{v}}^{b_{rn}(v)} F^{n-1}(v_{rn}(b)) \left[u'(v_{rn}(b) - b) - \frac{u(v_{rn}(b) - b)}{v_{rn}(b) - b} \right] db \right) \right. \\
&\quad \left. + F^{n-1}(v) u(v - b_{rn}(v)) \right\} f(v) dv. \tag{20}
\end{aligned}$$

In order to simplify the expression for G we first note that

$$\begin{aligned}
A := & \int_{\underline{v}}^{b_{rn}(v)} F^{n-1}(v_{rn}(b)) \left[-\frac{u(v_{rn}(b) - b)}{v_{rn}(b) - b} \right] db \stackrel{\text{parts}}{=} \\
& - u(v_{rn}(b) - b) \int_{\underline{v}}^x F^{n-1}(v_{rn}(b)) \frac{1}{v_{rn}(b) - b} db \Big|_{\underline{v}}^{b_{rn}(v)} + \\
& \int_{\underline{v}}^{b_{rn}(v)} \left(\int_{\underline{v}}^b F^{n-1}(v_{rn}(s)) \frac{1}{v_{rn}(s) - s} ds \right) u'(v_{rn}(b) - b) [v'_{rn}(b) - 1] db.
\end{aligned}$$

From (3) we have that in a first-price auction

$$\frac{1}{v_{rn}(b) - b} = \frac{F^{n-1}(v_{rn}(b))}{\int_{\underline{v}}^{v_{rn}(b)} F^{n-1}(s) ds}.$$

Using this relation and (19), we have that

$$\begin{aligned} \int_{\underline{v}}^b F^{n-1}(v_{rn}(s)) \frac{1}{v_{rn}(s) - s} ds &= \int_{\underline{v}}^b F^{n-1}(v_{rn}(s)) \frac{F^{n-1}(v_{rn}(s))}{\int_{\underline{v}}^{v_{rn}(s)} F^{n-1}(x) dx} ds \\ &= \int_{\underline{v}}^b (n-1) F^{n-2}(v_{rn}(s)) f(v_{rn}(s)) v'_{rn}(s) ds = F^{n-1}(v_{rn}(b)). \end{aligned}$$

Substituting this relation in the expression for A gives,

$$A = -u(v - b_{rn}(v)) F^{n-1}(v) + \int_{\underline{v}}^{b_{rn}(v)} F^{n-1}(v_{rn}(b)) u'(v_{rn}(b) - b) [v'_{rn}(b) - 1] db.$$

Substituting this relation in (20), we have

$$\begin{aligned} G &= \int_{\underline{v}}^{\bar{v}} \left(\int_{\underline{v}}^{b_{rn}(v)} F^{n-1}(v_{rn}(b)) u'(v_{rn}(b) - b) v'_{rn}(b) db \right) f(v) dv \\ &= \int_{\underline{v}}^{\bar{v}} \left(\int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) ds \right) f(v) dv \\ &= \left(\int_{\underline{v}}^v F^{n-1}(s) u'(s - b_{rn}(s)) ds \right) F(v) \Big|_{\underline{v}}^{\bar{v}} - \int_{\underline{v}}^{\bar{v}} F^n(v) u'(v - b_{rn}(v)) dv \\ &= \int_{\underline{v}}^{\bar{v}} F^{n-1}(v) (1 - F(v)) u'(v - b_{rn}(v)) dv. \end{aligned}$$

D Proof of Proposition 3

Similarly to the first-price auction, we can write the equilibrium bid as $v(b) = v_{rn}(b) + \varepsilon v_1(b) + O(\varepsilon^2)$, where $v_{rn}(b)$ is the inverse function of the risk-neutral equilibrium strategy in all-pay auctions (9). Note that, for clarity, we drop the superscript *all*. We first note

that when $\varepsilon \ll 1$,

$$F(v(b)) = F(v_{rn}) + \varepsilon v_1 F'(v_{rn}) + O(\varepsilon^2),$$

$$f(v(b)) = f(v_{rn}) + \varepsilon v_1 f'(v_{rn}) + O(\varepsilon^2),$$

$$\begin{aligned} U(v(b) - b) - U(-b) &= v(b) + \varepsilon[u(v(b) - b) - u(-b)] \\ &= v_{rn}(b) + \varepsilon[v_1(b) + u(v_{rn}(b) - b) - u(-b)] + O(\varepsilon^2), \end{aligned}$$

$$U'(v(b) - b) - U'(-b) = \varepsilon[u'(v(b) - b) - u'(-b)] = \varepsilon[u'(v_{rn}(b) - b) - u'(-b)] + O(\varepsilon^2).$$

Substitution in (8) and expanding in ε , the equation for the $O(1)$ is identical to the risk-neutral case and thus, automatically satisfied. The equation for the $O(\varepsilon)$ terms is

$$\begin{aligned} v_1'(b) &= \frac{F(v_{rn}(b))[u'(v_{rn}(b) - b) - u'(-b)]}{(n-1)f(v_{rn}(b))v_{rn}(b)} + \frac{u'(-b)}{(n-1)F^{n-2}(v_{rn}(b))f(v_{rn}(b))v_{rn}(b)} \\ &\quad - \frac{(n-2)v_1(b)}{(n-1)F^{n-1}(v_{rn}(b))v_{rn}(b)} - \frac{v_1 f'(v_{rn}(b))}{(n-1)F^{n-2}(v_{rn}(b))f^2(v_{rn}(b))v_{rn}(b)} \\ &\quad - \frac{[v_1(b) + u(v_{rn}(b) - b) - u(-b)]}{(n-1)F^{n-2}(v_{rn}(b))f(v_{rn}(b))v_{rn}^2(b)}, \end{aligned}$$

subject to $v_1(0) = 0$. Since, by (8),

$$v_{rn}'(b) = \frac{1}{(n-1)F^{n-2}(v_{rn}(b))f(v_{rn}(b))v_{rn}(b)}, \quad (21)$$

the equation for $v_1'(b)$ can be rewritten as

$$v_1'(b) + v_1(b)B(b) = G(b) \quad (22)$$

where

$$B(b) = \left[\frac{v_{rn}'(b)}{v_{rn}(b)} + \frac{f'(v_{rn}(b))}{f(v_{rn}(b))} v_{rn}'(b) + (n-2) \frac{f(v_{rn}(b))}{F(v_{rn}(b))} v_{rn}'(b) \right],$$

and

$$G(b) = v'_{rn}(b) \left\{ - \left[u(v_{rn}(b) - b) - u(-b) \right] (n-1) F^{n-2}(v_{rn}(b)) f(v_{rn}(b)) v'_{rn}(b) \right. \\ \left. + F^{n-1}(v_{rn}(b)) \left(u'(v_{rn}(b) - b) - u'(-b) \right) + u'(-b) \right\}. \quad (23)$$

The solution of (22) is given by

$$v_1(b) = e^{\int_b^{\bar{b}_{rn}} B} \left(C_1 - \int_b^{\bar{b}_{rn}} G(x) e^{-\int_x^{\bar{b}_{rn}} B} dx \right),$$

where $\bar{b}_{rn} = b_{rn}(\bar{v})$. It is easy to verify that (see (21))

$$e^{\int_b^{\bar{b}_{rn}} B} = \frac{v'_{rn}(b)}{v'_{rn}(\bar{b}_{rn})}.$$

Thus, as $b \rightarrow 0$, $v_{rn}(b) \rightarrow \underline{v}$ and $e^{\int_b^{\bar{b}_{rn}} B} \rightarrow \infty$. Therefore it follows that $C_1 = \int_0^{\bar{b}_{rn}} G(x) e^{-\int_x^{\bar{b}_{rn}} B} dx$

and thus,

$$v_1(b) = v'_{rn}(b) \int_0^b G(x)/v'_{rn}(x) dx.$$

Since $b_1(v) = -v_1/v'_{rn}(b)$ (see (15)) we get that

$$b_1(v) = - \int_0^{b_{rn}(v)} G(x)/v'_{rn}(x) dx.$$

Substitution of G from (23) gives

$$b_1(v) = \int_0^{b_{rn}(v)} \left\{ \left[u(v_{rn}(b) - b) - u(-b) \right] (F^{n-1}(v_{rn}(b)))' \right. \\ \left. - F^{n-1}(v_{rn}(b)) \left(u'(v_{rn}(b) - b) - u'(-b) \right) - u'(-b) \right\} db.$$

A few more technical calculations completes the proof.

E Proof of Proposition 4

In the case of weak risk aversion, the ex-ante expected utility for the buyers in equilibrium is given by

$$V^{all} = \int_{\underline{v}}^{\bar{v}} \left\{ F^{n-1}(v)v - b(v) + \varepsilon \left[F^{n-1}(v) (u(v - b(v)) - u(-b(v))) + u(-b(v)) \right] \right\} f(v) dv.$$

Using the relation $b(v) = b_{rn}(v) + \varepsilon b_1(v) + O(\varepsilon)$, we have

$$V^{all} = V_{rn}^{all} - \varepsilon \int_{\underline{v}}^{\bar{v}} \left\{ b_1(v) - \left[F^{n-1}(v) (u(v - b_{rn}(v)) - u(-b_{rn}(v))) + u(-b_{rn}(v)) \right] \right\} f(v) dv + O(\varepsilon^2).$$

Substituting (10) in the last equation and rearranging yields the result.