Strategic Substitutes and Potential Games*

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Abstract

We show that games of strategic substitutes (or complements) with aggregation are "pseudo-potential" games. The upshot is that they possess Nash equilibria in pure strategies (NE), even if the strategy sets are not convex; and that various dynamic processes converge to NE.

Our notion of aggregation is quite general and enables us to take a unified view of several disparate models.

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1 Introduction

Economic theory is replete with examples of what have come to be called, after Bulow et al [2], games of strategic substitutes (STS) or strategic com-

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plements (STC). They cover phenomena ranging from[1] oligopolistic competition between firms, to the problem of the commons (Dasgupta and Heal [3]), to macroeconomic coordination failures (Diamond [5]), to new technology adoption (Katz and Shapiro [8]), to bank runs (Diamond and Dybvig [4]). The essential feature of these games is that, when his competitors turn more aggressive, an agent has incentive to become less so (for STS) or more so (for STC). A thorough investigation has already been made into STC games. For instance, the existence of Nash equilibrium[2] (NE) was proved, under very general conditions, in Milgrom and Roberts [9] and Vives [14]. But a comparable study is lacking for STS games (except for the case of two players where, reversing the order of one player's strategies, any STS game is converted into an STC game). One goal of our paper is to begin to redress the imbalance.

The question of the existence of NE is probably more difficult to resolve in STS than in STC games. A simple example might put the matter in perspective. Suppose each player has a finite set of strategies. As is common in the STS and STC scenarios, we shall take them to be totally ordered by their "aggression" levels (and w.l.o.g. embedded in the real line, with higher numbers connoting more aggression). Start with the profile of "lowest" strategies, and require all players to unilaterally deviate, whenever possible, to their best-replies to the profile. Iterate the process. The sequence of profiles so generated will be monotonically increasing in STC games. Thus it will converge to an NE in finitely many steps. No similar argument can be made apriori for STS games. It also does not seem feasible to transport the general techniques and insights developed for STC games to the study of STS games.

In this paper we focus attention on STS games which have one further property: the payoff of a player depends only upon his own strategy, and some kind of "market aggregate" of others' strategies. This property is quite common to many examples, including the most famous of all STS games: Cournot oligopoly. In the Cournot setting, and indeed in many others, it suffices to take the aggregate to be just the sum of agents' actions. However there are situations (see Examples 1, 2, 3) when the strategic interaction between players is more complex, and simple additive aggregation will not

[1] This list is only meant to be indicative.

[2] Throughout, we confine ourselves to pure strategies; so NE will always mean "pure-strategy NE."

do the job. A broader concept of aggregation is called for, in order to bring out the hidden structure of the games and to render them amenable to our analysis. We motivate and develop such a concept in Section 5.

Once we have aggregation, a very striking thing occurs: in STS (or, for that matter, STC) games, players can be thought of as maximizing one *common* payoff function – the "pseudo"-potential – in order to deviate to a best reply (see Theorem 2). This is a variation on the more stringent notion of "ordinal" potential, set forth in Shapley [13] and elaborated in Monderer and Shapley [11], wherein *all* unilateral deviations had to be rank-ordered by the potential. We stress that the pseudo-potential is considerably less faithful to the game than the ordinal potential. Its maximization yields only a subset of the best replies, and in general it may rank unilateral deviations differently from their concomitant gains in payoff. Nevertheless the pseudo-potential turns out to be an effective technical tool for us.

The existence of a pseudo-potential for STS games with aggregation has some important ramifications. First and foremost, an NE always exists provided only that strategy sets are compact[3] (see Theorem 1 and Corollary 1). Furthermore, if the games have finite strategy sets then, for generic payoffs, sequential best-replies converge to NE (see Remark 1). This fact is obviously of use in computing NE, though we do not pursue that line of inquiry here. Finally, when strategy sets are convex and best replies are unique, the limit points of certain adaptive processes with simultaneous best replies (reminiscent of fictitious play) are also NE (see Remark 2).

Since our approach via the pseudo-potential relies only on compactness of strategy sets, it is especially suited to handle non-convexities that are bound to arise when indivisibilities are present in the underlying economic model. To highlight this point, we re-examine Cournot oligopoly in the quite general setting of Amir [1] (or, alternatively, of Novshek [12]). But we extend their models by dropping the hypothesis (maintained by both) that firms' strategy sets are convex. It turns out that we get an STS game with aggregation, and hence the NE exists (by our Theorem 5), generalizing the results of [1] and [12]. In particular, NE exist in the "discrete" Cournot model, where each firm can produce only finitely many levels of output (a fact already observed by Shapley in [13], when demand and costs are linear).

The paper is organized as follows. In Section 2 we introduce the notion of

[3]For STC games, the result follows from Milgrom and Roberts [9] and Vives [14], even without any appeal to aggregation.

STS games with simple (additive) aggregation, and state Theorem 1 on the existence of NE. Pseudo-potential games are introduced in Section 3. It is shown in Theorems 2 and 3 that pseudo-potential games include STS games, and always possess NE. Section 4 contains proofs of the theorems. We develop the concept of general (non-additive) aggregation in Section 5, and verify that our results remain intact. This implies in particular that our analysis can be carried over en toto to STC games (see Remark 4). In Section 6, we extend our approach to include discontinuous reaction functions (see Theorem 4 and Corollary 1). Finally, in Section 7, we show that indivisibilities in production do not disturb the existence of NE in the Cournot oligopoly model (see Theorem 5).

2 Strategic Substitutes with Aggregation

Consider a set of players $N = \{1, 2, ...n\}$. Each $i \in N$ has a set of strategies S^i, which is a compact subset of R_+. We assume, without loss of generality, that each S^i contains at least two distinct elements. Put $S \equiv S^1 \times ... \times S^n$. For any $s = (s^1, ..., s^n) \in S$ and $t \in S^i$, denote $(s^1, ..., s^{i-1}, t, s^{i+1}, ..., s^n)$ by $(s \mid_i t)$; $(s^1, ..., s^{i-1}, s^{i+1}, ..., s^n)$ by s_{-i}; and $\sum_{j \in N \setminus \{i\}} s^j$ by $\overline{s}_{-i}$. The payoff function $\pi^i : S \to R$ of player i depends only upon his own strategy s^i and the aggregate[4] $\overline{s}_{-i}$ of others' strategies. So, with a slight abuse of notation, we will write $\pi^i(s^i, \overline{s}_{-i})$ for $\pi^i(s)$, and view π^i as defined on the domain $S^i \times \overline{S}_{-i}$, where $\overline{S}_{-i} \equiv \sum_{j \in N \setminus \{i\}} S^j$.

For any choice $s_{-i} \in \prod_{j \in N \setminus \{i\}} S^j$ of others' strategies, the set $\beta^i(\overline{s}_{-i})$ of *best replies* of player i is given by

$$\beta^i(\overline{s}_{-i}) = \arg\max_{t \in S^i} \pi^i(t, \overline{s}_{-i}).$$

Recall that $s = (s^1, ..., s^n) \in S$ is a *Nash equilibrium* (NE) if

$$s^i \in \beta^i(\overline{s}_{-i})$$

for all $i \in N$.

Finally, let us recall the notion of strategic substitutes. We shall present it in a slightly more general form than is standard[5], allowing for multiplicity

[4]For a more general notion of aggregation see Section 5.

[5]The term "strategic substitutes" was introduced by Bulow et al [2] to refer to games in which the best reply functions of the players are downward sloping. (See also Fudenberg and Tirole [6].)

of best replies.

We say that $\Gamma = (N, S^1, ..., S^n, \pi^1, ..., \pi^n)$ is a game of *strategic substitutes* (STS) with aggregation if, for every $i \in N$, there exists a continuous[6] and nonincreasing best-reply selection $b^i : \overline{S}_{-i} \to S^i$, i.e.,

(i) $b^i(x) \in \beta^i(x)$ for all $x \in \overline{S}_{-i}$,

(ii) b^i is continuous on $\overline{S}_{-i}$,

and

(iii) $b^i(x) \leq b^i(y)$ whenever $x > y$.

Our main result is

Theorem 1. *A game of strategic substitutes with aggregation has a Nash equilibrium.*

Let us immediately remark that, if we define strategic complements (STC) exactly as STS above, except for replacing "$x > y$" by "$x < y$" in (iii), then Theorem 1 will hold for STC games with aggregation, as indeed will the rest of our analysis. (See Remark 4 for an explanation). But since existence results are already known for STC games, we keep STS games in the forefront.

3 Pseudo-Potential Games

We develop the notion of a pseudo-potential game, which will be crucial for establishing Theorem 1.

Consider a game $\widetilde{\Gamma} = (N, S^1, ..., S^n, \widetilde{\pi}^1, ..., \widetilde{\pi}^n)$ in which the players and their strategy-sets are as before, but payoff functions $\widetilde{\pi}^i : S \to R$ are allowed to take a general form. We say that $\widetilde{\Gamma}$ is a *pseudo-potential game* if there exists a continuous function $P : S \to R$ such that, for all $i \in N$ and all $s \in S$,

$$\arg\max_{t \in S^i} \widetilde{\pi}^i(s \mid_i t) \supset \arg\max_{t \in S^i} P(s \mid_i t). \tag{1}$$

In other words, each player's best reply correspondence in the game $\Gamma^* = (N, S^1, ..., S^n, P, ..., P)$ is included in that of $\widetilde{\Gamma}$: it suffices for a player to maximize the potential P, rather than his real payoff $\widetilde{\pi}^i$, in order to get to *some* best reply. One may therefore think of the potential P as a convenient

[6] If S^i is finite, the requirement of continuity is vacuous. For relaxations of continuity, see Section 6.

common proxy for all the different payoff functions $\widetilde{\pi}^i$, $i \in N$, in the analysis of NE of $\widetilde{\Gamma}$. (For, as is evident, NE of Γ^* are a fortiori NE of $\widetilde{\Gamma}$.)

The following two results immediately imply Theorem 1, but may be of independent interest.

Theorem 2. *A game of strategic substitutes with aggregation is a pseudo-potential game.*

Theorem 3. *A pseudo-potential game has a pure strategy NE.*

Remark 1 (Generic Convergence of Sequential Best Replies) In the light of Theorem 2, this remark is obvious, but seems worth putting on record. Consider an STS game with aggregation and with finite strategy sets. Then, for generic payoffs, all best reply correspondences will be single-valued. Assume this is the case. Start with an arbitrary strategy profile, and let each player, one at a time, unilaterally deviate to his unique best reply, if he does not happen to be there already. More precisely, let $\{s(t)\}_{t=1}^{\infty} = \{(s^1(t), ..., s^n(t))\}_{t=1}^{\infty}$ be a sequence of strategy-profiles with the following property. At every stage t, if $s^j(t-1) \neq \beta^j(\overline{s}_{-j}(t-1))$ for a nonempty set of players j, one of them (say i) is required to choose his best reply to $s_{-i}(t-1)$ at stage t, i.e., $s^i(t) = \beta^i(\overline{s}_{-i}(t-1))$; all the other players stay put at their previous strategies, i.e., $s_{-i}(t) = s_{-i}(t-1)$. Then the sequence becomes stationary in finite time, since the game is pseudo-potential, and since the monotone sequence $\{P(s(t))\}_{t=1}^{\infty}$ cannot have infinitely many strict increases on its finite domain S. The stationary profile is clearly an NE of $\widetilde{\Gamma}$.■

4 Proofs

4.1 Proof of Theorem 2

Let $\Gamma = (N, S^1, ..., S^n, \pi^1, ..., \pi^n)$ be an STS game with aggregation; and, for all $i \in N$, let $b^i : \overline{S}_{-i} \to S^i$ be a continuous and nonincreasing best-reply selection.

Denote by Σ_{-i} the convex hull of $\overline{S}_{-i}$, which is obviously compact. We extend b^i, in a piecewise-linear fashion, to a function τ^i, defined on the entire

domain Σ_{-i}. (Thus τ^i coincides with b^i on $\overline{S}_{-i}$, and if $x, y \in \overline{S}_{-i}$ are such that $(x, y) \subset \Sigma_{-i} \backslash \overline{S}_{-i}$, then τ^i is an affine function on $[x, y]$.) Furthermore, we enhance the domain of τ^i to include the interval $[-1, \min \Sigma_{-i}]$, by setting: $\tau^i(-1) = \max S^i$, $\tau^i(\min \Sigma_{-i}) = b^i(\min \Sigma_{-i})$, and then extending τ^i linearly on $(-1, \min \Sigma_{-i})$. Figure 1 illustrates the construction of τ^i in the case where[7] each S^i is a finite union of closed intervals (the bold line represents b^i and the broken line represents the rest of τ^i; bold intervals and dots on the vertical and horizontal axes represent the sets S^i, $\overline{S}_{-i}$ respectively).

Insert Figure 1

Notice that τ^i inherits continuity, and the property of being nonincreasing, from b^i. For every $i \in N$, now define $F_i : S^i \to R$ by

$$F_i\left(s^i\right) = \int_{-1}^{\max(\Sigma_{-i})} \min(\tau^i(x), s^i) dx.$$

Consider the continuous function[8] $P : S^1 \times ... \times S^n \to R$ given by

$$P\left(s^1, ..., s^n\right) = -\sum_i s^i - \sum_{i<j} s^i s^j + \sum_i F_i\left(s^i\right). \tag{2}$$

We claim that P renders Γ into a pseudo-potential game. To check this, fix $s \in S$, and suppose that $s^i \in \arg\max_{t \in S^i} P(s \mid_i t)$. Note that for any $t \in S^i$

$$P(s \mid_i t) = \left[-t\left(\overline{s}_{-i} + 1\right) + F_i(t)\right] + \left[-\sum_{j \neq i} s^j - \sum_{\substack{j<k \\ j,k \neq i}} s^j s^k + \sum_{j \neq i} F_j\left(s^j\right)\right]. \tag{3}$$

Since the second (bracketed) term in (3) is not a function of t, the first term is maximized at s^i (for the given s_{-i}). We will deduce from this that $s^i = \tau^i\left(\overline{s}_{-i}\right)$ $\left(= b^i\left(\overline{s}_{-i}\right)\right)$.

[7] In the case of a general compact set, there would be countably many gaps to be plugged between $\overline{S}_{-i}$ and Σ_{-i}.

[8] A function of similar form first came to our attention in Huang [7]. He, however, defined it under more restrictive assumptions on best replies, in the context of certain Cournot oligopoly games with convex strategy sets, in order to study properties of fictitious play.

Note first that if $t \leq \tau^i(\overline{s}_{-i})$, then the first term in (3) is equal to the area $A(t)$ of the region (shown shaded in Figure 2) which is bounded by: the graph of τ^i and the horizontal line $y = t$ from above,[9] the horizontal line $y = 0$ from below, the vertical line $x = \overline{s}_{-i}$ from the left, and the vertical line $x = \max \Sigma_{-i}$ from the right.

Insert Figure 2

On the other hand, if $t \geq \tau^i(\overline{s}_{-i})$ $(= b^i(\overline{s}_{-i}))$, then the first term in (3) is equal to the difference $A(b^i(\overline{s}_{-i})) - B(t)$, where $B(t)$ is the area of the region (shown shaded in Figure 3), which is bounded by: the graph of τ^i from below, the horizontal line $y = t$ from above, and the vertical line $x = \overline{s}_{-i}$ from the right.

Insert Figure 3

Since $A(t)$ is strictly increasing from $\min S^i$ to $\tau^i(\overline{s}_{-i})$, and since $B(t) > 0$ for $t > \tau^i(\overline{s}_{-i})$, we conclude that the first term in (3) has a unique maximum which is attained at $t = \tau^i(\overline{s}_{-i})$ $(= b^i(\overline{s}_{-i}))$, for the given s_{-i}. Thus,

$$\{s^i\} = \arg\max_{t \in \Sigma^i} P(s \mid_i t) = \{b^i(\overline{s}_{-i})\}.$$

Therefore b^i is indeed the best-reply (single-valued) correspondence of i in the game $(N, S^1, ..., S^n, P, ..., P)$. Since b^i, to begin with, was a selection from the best reply correspondence of $\Gamma = (N, S^1, ..., S^n, \pi^1, ..., \pi^n)$, we conclude that Γ is a pseudo-potential game.■

Remark 2 (Convergence of Adaptive Processes with Simultaneous Best Replies) Consider an STS game with aggregation and with convex strategy sets, and assume that all best reply correspondences are

[9] Where x and y denote the horizontal and vertical coordinates as usual.

single-valued. Suppose that the game is repeated, and that, in every period, players simultaneously choose the best reply to their *conjectures* of others' strategies. Unlike Remark 1, we now assume that players' conjectures take into account strategies observed in all previous periods, and not just the last one. Formally, let $\{\lambda_t\}_{t=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{t=1}^{\infty}\lambda_t = \infty$ and $\sum_{t=1}^{\infty}\lambda_t^2 < \infty$. (Here λ_t represents the weight given to the most recent observation in period t, or the speed with which players adapt their conjectures). Let $\{s(t)\}_{t=1}^{\infty} = \{(s^1(t), ..., s^n(t))\}_{t=1}^{\infty}$ be a sequence of strategy-profiles with the following property. At every period t, $s^i(t)$ is the unique best reply of i to his conjecture $\sigma_{-i}(t)$ of the true strategies of other players. Conjectures are defined recursively: $\sigma_{-i}(1)$ are arbitrary, and for all $j \in N\backslash\{i\}$

$$\sigma^j(t) = \lambda_t s^j(t-1) + (1-\lambda_t)\sigma^j(t-1).$$

(Setting $\lambda_t = \frac{1}{t}$ is reminiscent of fictitious play).

We claim that all limit points of the sequence $\{s(t)\}_{t=1}^{\infty}$ are NE. Indeed, this was established by Huang [7] for "best-reply potential games" (i.e., those games in which equality holds in place of inclusion in (1)), provided the potential is continuous and "multi-concave" (i.e., concave separately in each player's unilateral deviations). In our case, the assumption that each best reply is unique implies (via Theorem 2) that any STS game with aggregation is a best-reply potential game. Moreover, it is clear from (3) that multi-concavity of the potential P is tantamount to concavity of the functions F_i. This, in turn, follows from the fact that each τ^i is nonincreasing.■

4.2 Proof of Theorem 3

Let $\widetilde{\Gamma} = (N, S^1, ..., S^n, \widetilde{\pi}^1, ..., \widetilde{\pi}^n)$ be a pseudo-potential game with potential P. Suppose $s = (s^1, ..., s^n) \in \arg\max_{(t^1,...,t^n)\in S} P(t^1, ..., t^n)$ (such an s exists because P is continuous and S is compact). If s is not an NE of $\Gamma^* = (N, S^1, ..., S^n, P, ..., P)$, then $P(s \mid_i t) > P(s)$ for some $t \in S^i$, contradicting that s maximizes P.

But any NE of Γ^* is a fortiori an NE of $\widetilde{\Gamma}$, since best replies in Γ^* are by definition best replies in $\widetilde{\Gamma}$.■

5 Non-Additive Aggregation

Adding up players' strategies is but one way of aggregating them. It suits most of the examples we have cited, starting with Cournot oligopoly. However, there are also many kinds of strategic interaction which, at first glance, look alien to our framework. It is only when appropriate aggregators $\alpha : S_{-i} \to R$ are constructed for them, that their hidden structure is unmasked and they fit into our framework, with $\overline{s}_{-i}$ replaced by $\alpha(s_{-i})$ and $\overline{S}_{-i}$ by $\alpha(S_{-i})$.

Before giving examples, let us define our class of aggregators. Denote

$$s^*_{-i}(k) \equiv \sum_{\substack{i_1 < i_2 < \ldots < i_k \\ i_1, i_2, \ldots, i_k \neq i}} s_{i_1} \ldots s_{i_k}$$

for $1 \leq k \leq n-1$ (i.e., $s^*_{-i}(k)$ is the sum of all possible products of k distinct strategies picked from $s_{-i} = (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n)$; and so for $k = 1$ we get $s^*_{-i}(1) = \overline{s}_{-i}$.) Let $a_1, \ldots, a_{n-1}$ be scalars (not all of them zero), and define

$$\alpha(s_{-i}) \equiv \sum_{k=1}^{n-1} a_k s^*_{-i}(k).$$

For the moment assume, by way of simplicity, that the scalars a_k are such that $\alpha(s_{-i}) \geq 0$ for all $i \in N$ and all $s_{-i} \in S_{-i}$. (This restriction can be dropped, see Remark 2.) Notice that the aggregator $\alpha(s_{-i})$ is the *same* linear combination of $\left\{s^*_{-i}(k)\right\}_{k=1}^{n-1}$ for all $i \in N$.

Our aggregators are seemingly abstruse. We shall now give three examples to illustrate how they might arise in a natural manner. In all the examples, each player i chooses effort level $s^i \in [0, B^i]$ to apply to the personal task faced by him. This gives rise to the probability $p_i(s^i)$ of "success" in his task, where $p_i : [0, B^i] \to [0, 1]$ is a continuous and strictly increasing function with $p_i(0) = 0$ and $p_i(B)_i = 1$. By relabeling effort levels if necessary, we take $B^i = 1$ and $p_i(s^i) = s^i$. The events of individual success are assumed to be independent across different players. Furthermore, for ease of calculation, we suppose there are three players ($N = \{1, 2, 3\}$), and that each $i \in N$ incurs quadratic cost $c_i (s^i)^2$, on account of his effort s^i, for some constant $c_i > 0$.

Example 1 (Team Projects with Complementary Tasks) Each player's task is critical to the success of the team's project. Thus $s^1 s^2 s^3$ is

the probability that the project will succeed. Suppose $r_i > 0$ is the utility to player i of a successful project. This yields the payoff function

$$\pi^i\left(s^1, s^2, s^3\right) = r_i s^1 s^2 s^3 - c_i\left(s^i\right)^2 = r_i s^i \alpha\left(s_{-i}\right) - c_i\left(s^i\right)^2,$$

where $\alpha\left(s_{-i}\right)$ is the aggregator $s^*_{-i}(2) = s^j s^k$ (and, $N\backslash\{i\} = \{j, k\}$). Then i's best reply is

$$\beta^i\left(\alpha\left(s_{-i}\right)\right) = \left\{\min\left(\frac{r_i \alpha\left(s_{-i}\right)}{2c_i}, 1\right)\right\},$$

which is a nondecreasing function of $\alpha\left(s_{-i}\right)$; and shows that we have an STC game with aggregation (when $\overline{s}_{-i}$ is replaced by $\alpha\left(s_{-i}\right)$).■

Example 2 (Team Projects with Substitutable Tasks) Here we suppose that each player by himself can make the project successful. Then the probability that the project is successful is

$$\begin{aligned} f\left(s^1, s^2, s^3\right) &= 1 - \left(1 - s^1\right)\left(1 - s^2\right)\left(1 - s^3\right) \\ &= s^1 + s^2 + s^3 - s^1 s^2 - s^1 s^3 - s^2 s^3 + s^1 s^2 s^3, \end{aligned}$$

and the payoff to player i is

$$\pi^i\left(s^1, s^2, s^3\right) = r_i f\left(s^1, s^2, s^3\right) - c_i\left(s^i\right)^2 = r_i s^i\left[1 - \alpha\left(s_{-i}\right)\right] + \alpha\left(s_{-i}\right) - c_i\left(s^i\right)^2,$$

where $\alpha\left(s_{-i}\right)$ is the aggregator $s^*_{-i}(1) - s^*_{-i}(2)$. Thus

$$\beta^i\left(\alpha\left(s_{-i}\right)\right) = \left\{\min\left(\frac{r_i\left[1 - \alpha\left(s_{-i}\right)\right]}{2c_i}, 1\right)\right\}$$

is a nonincreasing function of $\alpha\left(s_{-i}\right)$, and so this example describes an STS game with aggregator α.■

Example 3 (Tournaments) Assume that a reward of r dollars is shared by the group of players who succeed. If only one player succeeds, he gets r for sure; if exactly two succeed, each gets r with probability $\frac{1}{2}$; if all three succeed, each gets r with probability $\frac{1}{3}$. By rescaling utilities, we may assume

w.l.o.g. that r dollars yield r utiles to each player. Then the expected value of the reward to i is

$$rs^i\left(1-s^j\right)\left(1-s^k\right)+\frac{r}{2}s^is^j\left(1-s^k\right)+\frac{r}{2}s^is^k\left(1-s^j\right)+\frac{r}{3}s^is^js^k$$

$$=rs^i\left[1-\frac{1}{2}s^j-\frac{1}{2}s^k+\frac{1}{3}s^js^k\right]=rs^i\left[1-\alpha\left(s_{-i}\right)\right],$$

where $\alpha\left(s_{-i}\right)=\frac{1}{2}s^*_{-i}\left(1\right)-\frac{1}{3}s^*_{-i}\left(2\right)$. Therefore each player's payoff function is

$$\pi^i\left(s^1,s^2,s^3\right)=rs^i\left[1-\alpha\left(s_{-i}\right)\right]-c_i\left(s^i\right)^2.$$

Consequently,

$$\beta^i\left(\alpha\left(s_{-i}\right)\right)=\left\{\min\left(\frac{r\left[1-\alpha\left(s_{-i}\right)\right]}{2c_i},1\right)\right\}$$

is a nonincreasing function of $\alpha\left(s_{-i}\right)$, and therefore tournaments are also STS games with aggregator α.■

As we said, our results remain intact if we postulate that the payoff to any player i depends only upon his own strategy s^i and the aggregate $\alpha\left(s_{-i}\right)$ of others' strategies. Other than the obvious change of notation ($\overline{s}_{-i}$ replaced by $\alpha\left(s_{-i}\right)$ and $\overline{S}_{-i}$ by $\alpha\left(S_{-i}\right)$), the only variation needed is in the proof of Theorem 2. We redefine P (which was defined for the additive aggregator in (2)) as follows:

$$P\left(s^1,...,s^n\right)=-\sum_i s^i-\sum_{k=1}^{n-1}a_k\cdot\sum_{i_1<i_2<...<i_{k+1}}s^{i_1}...s^{i_{k+1}}+\sum_i F_i\left(s^i\right).$$

Note that for any $i\in N$,

$$P\left(s\mid_i t\right)=\left[-t\left(\sum_{k=1}^{n-1}a_ks^*_{-i}(k)+1\right)+F_i\left(t\right)\right]+\left[-\sum_{j\neq i}s^j-\sum_{k=1}^{n-2}a_ks^*_{-i}(k+1)+\sum_{j\neq i}F_j\left(s^j\right)\right]$$

$$=\left[-t\left(\alpha\left(s_{-i}\right)+1\right)+F_i\left(t\right)\right]+\left[-\sum_{j\neq i}s^j-\sum_{k=1}^{n-2}a_ks^*_{-i}(k+1)+\sum_{j\neq i}F_j\left(s^j\right)\right].$$

The above equality replaces (3) in the proof of Theorem 2, and the rest of the arguments hold exactly as before.

Remark 3 (Aggregation without the Positivity Requirement) The requirement that aggregation be nonnegative can be dropped. If $\alpha(s_{-i})$ is negative for some $s_{-i} \in S_{-i}$, we can define another aggregator $\widetilde{\alpha}$ by $\widetilde{\alpha}(s_{-i}) \equiv \alpha(s_{-i}) + a$ for all $i \in N$ and all $s_{-i} \in S_{-i}$. Clearly, for large enough a, $\widetilde{\alpha}(s_{-i})$ is always nonnegative. It is obvious that if i's payoff is a function of s_i and $\alpha(s_{-i})$, then it is representable also as a function of s_i and $\widetilde{\alpha}(s_{-i})$, and any nonincreasing and continuous best-reply selection remains such after this change of variables. Our analysis holds with these "non-homogeneous"aggregations just as well. One only has to add $s^*_{-i}(0) \equiv 1$ to the set $\left\{s^*_{-i}(k)\right\}_{k=1}^{n-1}$, and allow aggregators $\alpha(s_{-i})$ to be linear combinations of $\left\{s^*_{-i}(k)\right\}_{k=0}^{n-1}$, not just $\left\{s^*_{-i}(k)\right\}_{k=1}^{n-1}$.■

Remark 4 (STC Games with Aggregation) The previous remark also enables us to include STC games with aggregation in our approach. Indeed, given an STC game with aggregator α, the payoff function of each player i can be obviously redefined to depend on s^i and $\widetilde{\alpha}(s_{-i}) \equiv -\alpha(s_{-i})$, instead of s^i and $\alpha(s_{-i})$. Consequently, if a best-reply selection b^i is a nondecreasing function of α (which is the case for STC), it turns into $\widetilde{b}^i(\widetilde{\alpha}) \equiv b^i(-\widetilde{\alpha})$, a nonincreasing function of $\widetilde{\alpha}$, and our analysis goes through by Remark 3. Note that this trick is purely technical, and does not change the STC character of the game: while $\widetilde{b}^i(\widetilde{\alpha}(s_{-i}))$is a nonincreasing function of $\widetilde{\alpha}(s_{-i})$, it remains nondecreasing in the underlying basic variable s_{-i}.■

6 Discontinuous Best Reply Selections

We do not know if continuity of our best reply selections is necessary for the validity of Theorem 1. However, we do have the following extension:

Theorem 4. *The conclusion of Theorem 1 remains intact, even without requiring continuity of the best reply selections, provided one of the following assumptions is made:*

(i) Γ is a game of strict strategic substitutes, i.e., for every $i \in N$ there exists a best reply selection b^i which is a strictly decreasing function[10] *of $\overline{s}_{-i}$;*

(ii) for every $i \in N$ there exists a best reply selection b^i which is nonincreasing and right-continuous in $\overline{s}_{-i}$;

(iii) for every $i \in N$ there exists a best reply selection b^i which is nonincreasing and left-continuous in $\overline{s}_{-i}$.

Proof. Given a nonincreasing best reply selection b^i for each player i, construct τ^i and P exactly as in the proof of Theorem 2. P is continuous as before, but this time

$$\arg\max_{t \in S^i} P(s \mid_i t) = \left[\lim_{x \downarrow \overline{s}_{-i}} \tau^i(x), \lim_{x \uparrow \overline{s}_{-i}} \tau^i(x)\right] \cap S^i \tag{4}$$

for all $i \in N$ and all $s \in S$. In particular, $\arg\max_{t \in S^i} P(s \mid_i t)$ need not be single-valued, if τ^i is discontinuous at $\overline{s}_{-i}$. (This is why P may fail to be a pseudo-potential function for the given game.) However, it follows from (4) that

$$b^i(\overline{s}_{-i}) \in \arg\max_{t \in S^i} P(s \mid_i t), \tag{5}$$

as before.

Now consider some $s = (s^1, ..., s^n) \in \arg\max_{(t^1,...,t^n) \in S} P(t^1, ..., t^n)$. Suppose first that assumption (i) is satisfied, and let b^i be a strictly decreasing best reply selection for each i. Note that if there is $i \in N$ (say, $i = 1$) such that $s^1 \neq b^1(\overline{s}_{-1})$, then by (5), $s' \equiv (b^1(\overline{s}_{-1}), s^2, ..., s^n) \in \arg\max_{(t^1,...,t^n) \in S} P(t^1, ..., t^n)$. Obviously, since s and s' maximize P, $s^i \in \arg\max_{t \in S^i} P(s \mid_i t)$ and $(s')^i \in \arg\max_{t \in S^i} P(s' \mid_i t)$ for all $i \in N$. Then (4) implies

$$s^2 \in \left[\lim_{x \downarrow \overline{s}_{-2}} \tau^2(x), \lim_{x \uparrow \overline{s}_{-2}} \tau^2(x)\right] \cap \left[\lim_{x \downarrow \overline{s'}_{-2}} \tau^2(x), \lim_{x \uparrow \overline{s'}_{-2}} \tau^2(x)\right].$$

But clearly $\overline{s}_{-2} \neq \overline{s'}_{-2}$, and so, from the fact that τ^2 is strictly decreasing, the intersection of the above two intervals must be empty. This is a contradiction, so $s^i = b^i(\overline{s}_{-i})$ for all $i \in N$, and s is an NE of Γ.

[10] For ease of notation, we revert from $\alpha(s_{-i})$ to $\overline{s}_{-i}$ (though the argument holds replacing $\overline{s}_{-i}$ by $\alpha(s_{-i})$ throughout, provided we assume that $\alpha(s_{-i})$ is *strictly* increasing in s_{-i}).

Next suppose that assumption (ii) holds, and let b^i be a nonincreasing and right-continuous best reply selection for each i. Then, since every τ^i is also nonincreasing and right-continuous, it follows from (4) that

$$b^i\left(\overline{s}_{-i}\right) = \min\left[\arg\max_{t\in S^i} P(s \mid_i t)\right] \tag{6}$$

for all $i \in N$. If (say) $s^1 \notin \arg\max_{t\in S^1} \pi^1\left(s^1, \overline{s}_{-1}\right)$, then $b^1\left(\overline{s}_{-1}\right) < s^1$ by (6) and the fact that $s^1 \in \arg\max_{t\in S^1} P(s \mid_1 t)$. Thus for all $j \neq 1$

$$\overline{s'}_{-j} < \overline{s}_{-j}, \tag{7}$$

where (recall) $s' \equiv \left(b^1\left(\overline{s}_{-1}\right), s^2, ..., s^n\right) \in \arg\max_{(t^1,...,t^n)\in S} P(t^1, ..., t^n)$. Since

$$s^j \in \arg\max_{t\in S^j} P(s \mid_j t) \cap \arg\max_{t\in S^j} P(s' \mid_j t)$$

for all $j \neq 1$, the conjunction of (4), (7), and the fact that τ^j is nonincreasing, yields

$$s^j = \min\left[\arg\max_{t\in S^j} P(s' \mid_j t)\right]. \tag{8}$$

The right-hand side of (8) is equal to $b^j\left(\overline{s'}_{-j}\right)$ by (6). Thus $\left(s'\right)^j = s^j = b^j\left(\overline{s'}_{-j}\right)$ for all $j \neq 1$. Since $\left(s'\right)^1 = b^1\left(\overline{s}_{-1}\right) = b^1\left(\overline{s'}_{-1}\right)$ by definition, s' is an NE of Γ.

Finally, when assumption (iii) holds, the analysis is similar as for assumption (ii).■

The following corollary of Theorem 4 will be used in the next section. It provides conditions on best reply correspondences, rather than their (possibly discontinuous) selections, that guarantee existence of NE in the game. Condition (c) expresses a somewhat stronger sense of "strategic substitutes" than what was used until now: the best reply correspondence itself is now required to be nonincreasing, not just *some* selection from it.

Corollary 1. *Assume that the best-reply correspondence β^i of every player i in Γ is:*

a) nonempty-valued;

b) upper hemi-continuous (that is, if $\{(x_n, y_n)\}_{n=1}^{\infty} \subset \overline{S}_{-i} \times S^i$ is such that $y_n \in \beta^i(x_n)$ and $\lim_{n\to\infty}(x_n, y_n) = (x, y)$, then $y \in \beta^i(x)$);

and

c) nonincreasing in the sense that[11] $\max \beta^i(x_1) \leq \min \beta^i(x_2)$ *whenever* $x_1 > x_2$.

Then Γ *has an NE.*

Proof. Note that

$$b_r^i(x) = \min \beta^i(x) \text{ for all } x \in \overline{S}_{-i}$$

defines a nonincreasing best-reply selection which is right-continuous, and

$$b_l^i(x) = \max \beta^i(x) \text{ for all } x \in \overline{S}_{-i}$$

defines a nonincreasing best-reply selection which is left-continuous. Thus, Γ satisfies both (ii) and (iii) of Theorem 4.■

7 Cournot Oligopoly with Indivisibilities

Consider Cournot oligopoly with the set $N = \{1, 2, ..., n\}$ of firms. Following Amir [1], we assume that the inverse demand function P is strictly decreasing and log-concave; the cost function c_i of each firm i is strictly increasing and left-continuous; and each firm's monopoly profit (i.e., $xP(x) - c_i(x)$) becomes negative for large enough x. However, unlike Amir [1], we allow for indivisibilities in production. The strategy sets S^i, consisting of all possible levels of output producible by firm i, are not required to be convex but just closed[12]. Thus, in particular, the "discrete" Cournot model, in which each firms's outputs consist of indivisible units, will be included in our analysis.

Theorem 5 below extends Theorem 3.1 of Amir [1], which took $S^i = R_+$. Its proof simply exploits the fact (essentially due to Amir [1]) that Cournot oligopoly is a game of strategic substitutes (in the sense expressed in (c) of Corollary 1).

Theorem 5. *Under the above assumptions, Cournot oligopoly has an NE.*

Proof. Amir [1] showed that the payoff function π^i of each firm i has the "dual strong single crossing property" (DSSCP):

$$\pi^i(x_1, y_2) \leq \pi^i(x_2, y_2) \Rightarrow \pi^i(x_1, y_1) < \pi^i(x_2, y_1),$$

[11] Note that $\max \beta^i(x)$ and $\min \beta^i(y)$ are well defined by upper hemi-continuity.

[12] We now drop the compactness (i.e., boundedness) requirement.

where[13] $\pi^i(x,y) = xP(x+y) - c_i(x)$, for *all* $x_1 > x_2 \geq 0$ and $y_1 > y_2 \geq 0$. Afortiori the DSSCP continues to hold when the strategy sets are closed *subsets* of R_+. It is then easy to verify (alternatively, use Theorem 4 of Milgrom and Shannon [10]) that each β^i is nonincreasing. Also, since profits become negative for large outputs, we may w.l.o.g. restrict the strategy set S^i of each firm i to be compact. Finally, notice that $\beta^i(y) \equiv \arg\max_{x \in S^i} \pi^i(x,y)$ is nonempty-valued and upper hemi-continuous in $y \in \overline{S}_{-i}$, since π^i is continuous in y and upper semi–continuous in (x,y). Thus, according to Corollary 1, NE exists.■

Remark 5 (Novshek's Existence Theorem for Cournot Oligopoly) In [12], Novshek established the existence of NE provided P is strictly decreasing and twice continuously differentiable, and satisfies

$$P'(x) + xP''(x) \leq 0, \tag{9}$$

for all x below the point where the inverse demand reaches zero. (The cost functions were required to be only weakly, not strictly, increasing). But, as was shown in the proof of Theorem 3 in [12], each β^i is still nonempty, upper hemi-continuous and nonincreasing. Thus, Novshek's existence result is also implied by Corollary 1.

To extend Novshek's model to allow for indivisibilities, we may replace (9) by its discrete version[14]:

$$x_1P(x_1+y_2) - x_2P(x_2+y_2) > x_1P(x_1+y_1) - x_2P(x_2+y_1)$$

for all $x_1 > x_2 \geq 0$ and $y_1 > y_2 \geq 0$. This clearly implies the DSSCP of the payoff functions, and so, just as in the previous scenario, our Corollary 1 yields the existence of NE even when the strategy sets S^i are not convex but merely compact.■

[13] $x \equiv$output of firm i, $y \equiv$aggregate output of all firms other than i.

[14] Condition (9) was shown in [12] to be equivalent to the fact that the marginal revenue of a firm is non-increasing in the aggregate output of other firms. Therefore, our condition is indeed a discrete version of (9), but with a strict, rather than weak, inequality.

References

1. Amir, R. (1996), "Cournot Oligopoly and the Theory of Supermodular Games," *Games and Economic Behavior* **15**, 132-148.

2. Bulow, J. I., J. D. Geanakoplos, and P. D. Klemperer (1985), "Multimarket Oligopoly: Strategic Substitutes and Complements," *Journal of Political Economy* **93**, 488-511.

3. Dasgupta, P., and G. Heal (1979), Economic Theory and Exhaustible Resources, Cambridge Economics Handbooks, Cambridge University Press.

4. Diamond, D. W. and P. Dybvig (1983), "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy* **91,** 401-419.

5. Diamond, P. (1982), "Aggregate Demand Management in Search Equilibrium," *Journal of Political Economy* **90**, 881-894.

6. Fudenberg, D. and J. Tirole (1986), Dynamic Models of Oligopoly, Chur, Switzerland, Hardwood Academic Publishers.

7. Huang, Z. (2002), "Fictitious Play for Games with a Continuum of Strategies," Ph.D. Dissertation, Department of Economics, SUNY at Stony Brook.

8. Katz, M., and C. Shapiro (1986), "Technology Adoption in the Presence of Network Externalities," *Journal of Political Economy* **94**, 822-841.

9. Milgrom, P., and J. Roberts (1990), "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica* **58**, 1255-1277.

10. Milgrom, P., and C. Shannon (1994), "Monotone Comparative Statics," *Econometrica* **62**, 157-180.

11. Monderer, D., and L. Shapley (1996), "Potential Games," *Games and Economic Behavior* **14,** 124-143.

12. Novshek, W. (1985), "On the Existence of Cournot Equilibrium," *Review of Economic Studies* **52**, 85-98.

13. Shapley, L. (1994), Lecture Notes in Game Theory, Handouts # 9 and 10, Eco 145, Department of Economics, UCLA.

14. Vives, X. (1990), "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics* **19**, 305-321.

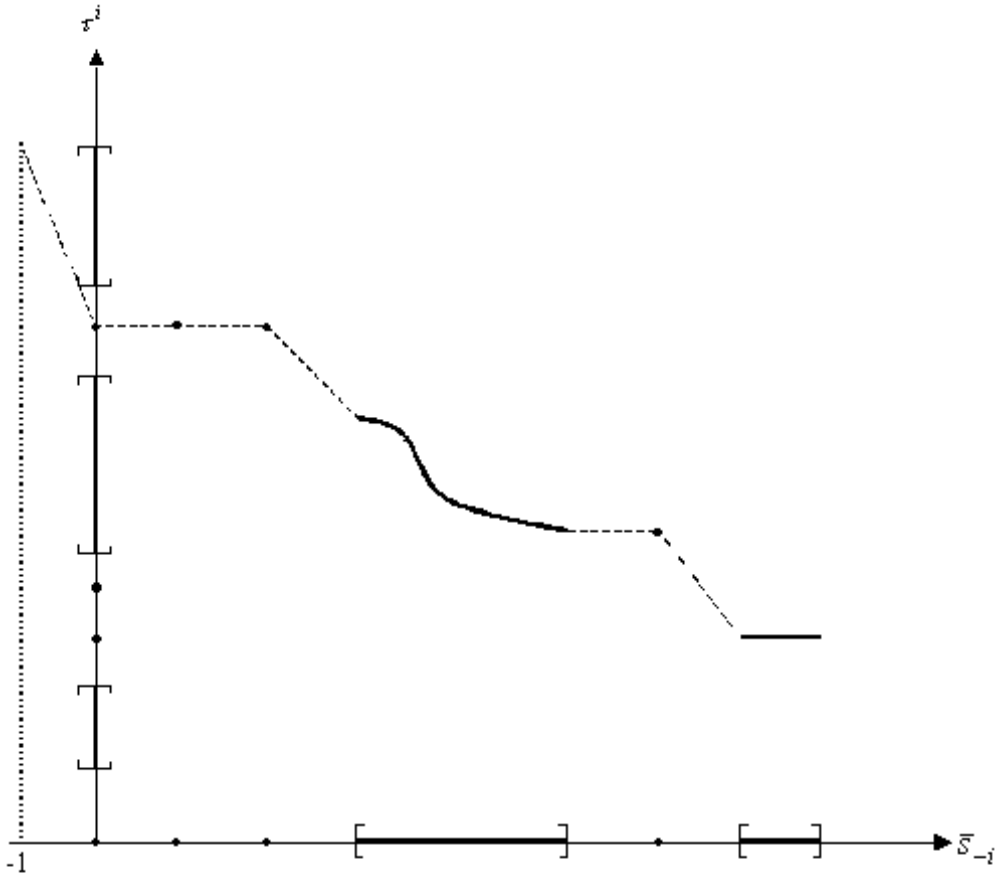
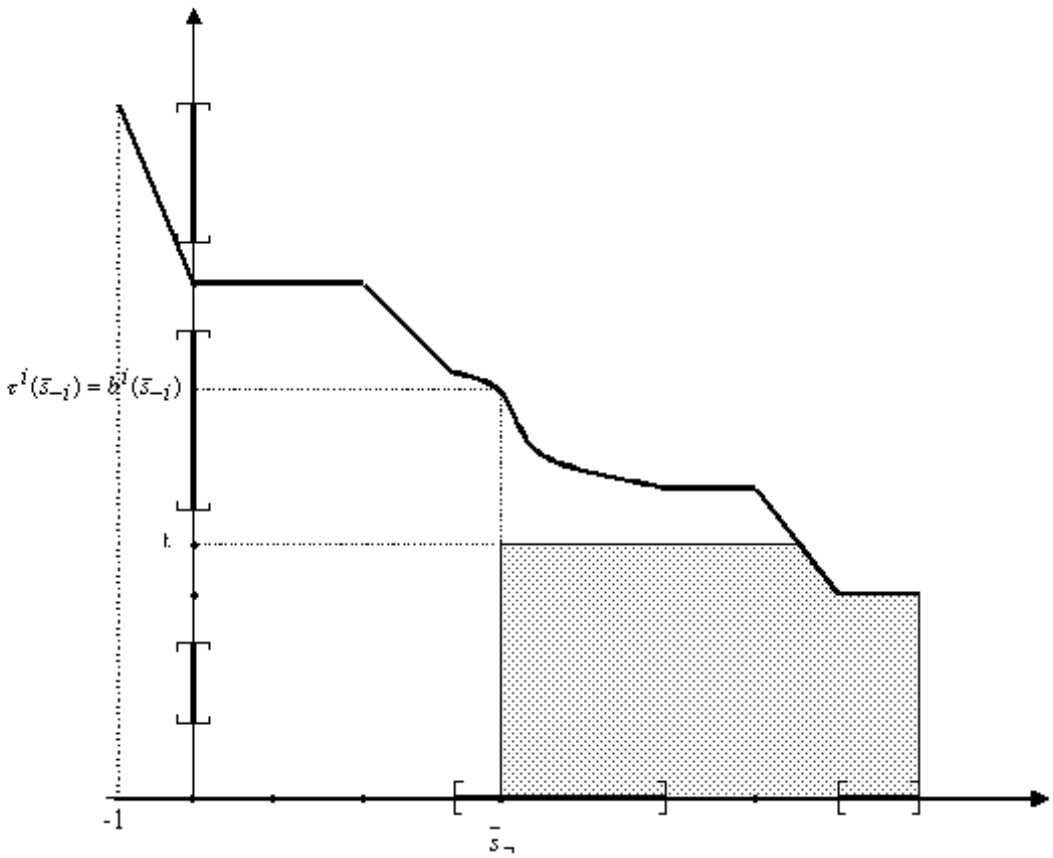


Figure 1:



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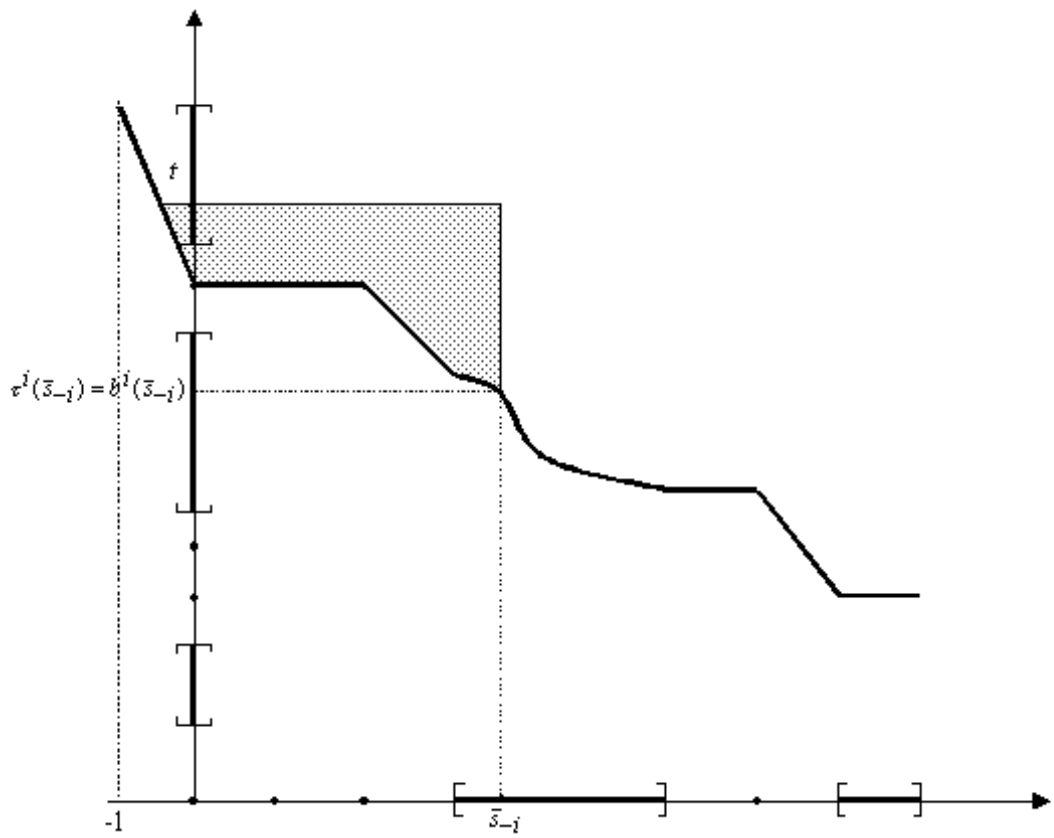

Figure 3: