Revenue Equivalence in Asymmetric Auctions

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Abstract

The Revenue Equivalence Theorem is generalized to the case of asymmetric auctions in which each player's valuation is independently drawn from a common support according to his/her distribution function as follows. Let ϵ be the level of asymmetry among the distribution functions and let $R(\epsilon)$ be the seller's expected revenue in equilibrium. Then, $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where both $R(0)$ and $R'(0)$ are independent of the auction mechanism.

Keywords: Asymmetric Auctions; Revenue Equivalence; Perturbation Analysis; Averaging.

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1 Introduction

A fundamental result in auction theory is the Revenue Equivalence Theorem, which states that the expected revenue of the seller in equilibrium is independent of the auction mechanism under quite general conditions. Vickrey (1961) established the revenue equivalence of the classical auction mechanisms (first-price auctions, Dutch auctions, English auctions, and second-price auctions). This result was generalized twenty years later by Myerson (1981), and independently by Riley and Samuelson (1981). Vickrey (1961) and Riley and Samuelson (1981) proved the revenue equivalence of *symmetric auctions* (auctions in which the valuations of all the players are drawn from the same distribution function). Myerson (1981) showed that the Revenue Equivalence Theorem remains true for *asymmetric auctions* (auctions in which the bidders' valuations are drawn independently from different distributions) provided that at any realization of the players' valuations the probability of a player to win the object is independent of the auction mechanism. This condition, however, does not usually hold in asymmetric auctions. Indeed, it is well known that asymmetric auctions are not necessarily revenue equivalent. For example, the expected revenue in first-price auctions can be higher or lower than in second-price auctions (see, e.g., Maskin and Riley (2000a)).

Auction theory has mostly dealt with symmetric auctions, since in this case there is an explicit expression for the equilibrium bidding strategies which can be used in the analysis. In many cases, however, bidders' valuations are drawn from different distribution functions. Because explicit expressions for asymmetric equilibrium strategies cannot be obtained except for very simple models, analysis of asymmetric auctions is considerably more complex and relatively little is known about them at present.

In this paper we study the seller's revenue in asymmetric auctions. Consider, for example, a situation where initially the distribution functions of all players are identical, but then the distribution function of each player undergoes a mild independent change. How does the seller's expected revenue change as a result of this weak asymmetry? Since there are no explicit solutions for the new equilibrium strategies, there is no exact answer to this question. In situations like this where it is difficult or even not possible to obtain exact solutions, much insight can be gained by employing *perturbation analysis*, whereby one calculates an approximation to the solution. As we shall see, such approximate solutions can be very insightful, making the sacrifice of 'exactness' worthwhile. Indeed, perturbation analysis is one of the most powerful tools in applied mathematics, and is extensively used in the analysis of mathematical models in the exact sciences.

In Section 2 we analyze the effect of weak asymmetry on the seller's expected revenue by using perturbation analysis. We show that under the same conditions as those of the classical (i.e., symmetric) Revenue Equivalence Theorem, the change of the seller's expected revenue as a result of weak asymmetry is, to leading order, independent of the form of the auction mechanism. Formally, let ϵ be the level of asymmetry among the distribution functions and $R(\epsilon)$ the seller's expected revenue in equilibrium. Then, we show that $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where both $R(0)$ (the seller's expected revenue in the symmetric case) and $R'(0)$ (the leading-order effect of the asymmetry) are independent of the auction mechanism. The value of $R''(0)$, however, does depend on the auction mechanism. Hence, strictly speaking, asymmetric auctions are not revenue equivalent, while weakly asymmetric auctions are ‘essentially’ revenue equivalent.

In Section 3 we implement a powerful concept from perturbation theory known as *averaging* to asymmetric auctions. We show that the seller’s expected revenue in asymmetric auctions can be well-approximated with the one in the symmetric case with the same number of players whose distribution function is the arithmetic average of the asymmetric distribution functions. Hence, when the asymmetry is weak, the revenues in the asymmetric case and in the corresponding symmetric case are essentially identical.

In Sections 4 and 5 the strength of our approach is illustrated by comparing our analytical results of the revenue in a first-price and second-price auction with the exact solutions obtained by numerical calculations. Although our results are derived under the assumption of weak asymmetry, this comparison suggests that these approximations remain accurate even when the asymmetry is not small.

2 The model

Throughout this paper we consider auction mechanisms with n players that satisfy the following conditions of the (symmetric) Revenue Equivalence Theorem (see Riley and Samuelson (1981)):

Condition 1 All players are risk neutral.

Condition 2 Player i’s valuation is private information to i and is independently drawn by a continuously differentiable distribution function $F_i(v)$ from a support $[\underline{v}, \bar{v}]$

which is common to all players.

Condition 3 The object is allocated to the player with the highest bid.

Condition 4 Any player with valuation $\underline{v}$ expects zero surplus.

Consider distribution functions of the form[1]

$$F_i(v) = F(v) + \epsilon H_i(v) \ , \qquad i = 1, \dots, n \ , \tag{1}$$

where $F(\underline{v}) = 0$, $F(\bar{v}) = 1$, $H_i(\underline{v}) = H_i(\bar{v}) = 0$ and $|H_i| \leq 1$ in $[\underline{v}\ , \overline{v}]$ for all i. Thus, ϵ *is a measure of the level of asymmetry.* We assume that when ε is sufficiently small (i.e., weak asymmetry) the auction has equilibrium bids $\{b_i(v;\epsilon)\}_{i=1}^n$ that are strictly increasing functions of v and continuously differentiable with respect to ε.[2]

Theorem 1 *Consider an auction mechanism that satisfies Conditions 1–4. Denote by $R(\epsilon)$ the seller's expected revenue as a function of ϵ when all bidders follow their equilibrium strategies. Then,*

$$R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2) \ ,$$

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[1] The assumption that the distribution functions are of the form (1) is not restrictive. Indeed, we can bring any family of distribution functions $\{F_i\}_{i=1}^n$ to this form by defining $F = \frac{1}{n}\sum_{i=1}^n F_i$, $\epsilon = \max_i \max_v |F_i - F|$ and $H_i = (F_i - F)/\epsilon$.

[2] Throughout this paper we study auction mechanisms for which an equilibrium does exist. Note that an equilibrium exists in asymmetric second-price auctions since the players have weak dominant strategies. Under mild assumptions an equilibrium exists also in asymmetric first-price auctions (Maskin and Riley (2000b), Lebrun (1996, 1999)).

where

$$R(0) = \bar{v} + (n-1)\int_{\underline{v}}^{\bar{v}} F^n(v)dv - n\int_{\underline{v}}^{\bar{v}} F^{n-1}(v)\,dv \ , \tag{2}$$

and

$$R'(0) = -(n-1)\int_{\underline{v}}^{\bar{v}} F^{n-2}(v)(1-F(v))\sum_{i=1}^{n} H_i(v)\,dv \ . \tag{3}$$

Remark. The classical Revenue Equivalence Theorem states that $R(0)$ is independent of the auction mechanism. Theorem 1 shows that $R'(0)$ is also independent of the auction mechanism. The value of $R''(0)$, however, does depend on the auction mechanism (see Section 5). Therefore, asymmetric auctions are 'essentially', but not strictly, revenue equivalent.

Proof of Theorem 1. Let $E_i(v)$, $S_i(v)$ and $P_i(v)$ be the expected payment, the expected surplus, and the probability of winning of bidder i with type v at equilibrium, respectively. Therefore,

$$S_i = vP_i(v) - E_i(v) \ . \tag{4}$$

It is well known (see, e.g., Klemperer (1999)) that

$$\frac{dS_i}{dv} = P_i(v) \ . \tag{5}$$

From (4),(5) it follows that

$$E_i'(v) = vP_i'(v) \ . \tag{6}$$

Let R_i be the expected payments of player i averaged across her types. Then,

$$\begin{aligned} R_i &= \int_{\underline{v}}^{\bar{v}} E_i(v)F_i'(v)\,dv = E_i(v)F_i\Big|_{\underline{v}}^{\bar{v}} - \int_{\underline{v}}^{\bar{v}} E_i'(v)F_i(v)\,dv \\ &= E_i(\bar{v}) - \int_{\underline{v}}^{\bar{v}} vP_i'(v)F_i(v)\,dv \ . \end{aligned}$$

Since $S_i(\underline{v}) = 0$ (Condition 4) we have from (6) that $E_i(\bar{v}) = \int_{\underline{v}}^{\bar{v}} vP_i'(v)\,dv$. Hence,

$$R_i = \int_{\underline{v}}^{\bar{v}} vP_i'(v)(1 - F_i(v))\,dv = -\int_{\underline{v}}^{\bar{v}} P_i(v)[v(1 - F_i(v))]'\,dv \ .$$

The seller's expected revenue is thus given by

$$R = \sum_{i=1}^{n} R_i = -\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} P_i(v)[v(1 - F_i(v))]'\,dv \ .$$

Now let F_i be given by (1). Then $R = R(\epsilon)$ depends on the asymmetry parameter ϵ. The Revenue Equivalence Theorem for symmetric auctions states that $R(0)$ is independent of the auction mechanism and is given by (2) (Riley and Samuelson (1981)). We now calculate $(dR/d\epsilon)_{\epsilon=0}$ and show, in particular, that it is also independent of the auction mechanism. Indeed,

$$\frac{dR}{d\epsilon}\bigg|_{\epsilon=0} = I_1 + I_2 \ , \tag{7}$$

where

$$I_1 = -\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} \frac{dP_i(v)}{d\epsilon}\bigg|_{\epsilon=0} [v(1 - F(v))]'\,dv \ , \quad I_2 = \sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} P(v)[vH_i(v)]'\,dv \ ,$$

and where $P(v) = F^{n-1}(v)$ is the probability of winning for a player with type v at equilibrium in the symmetric case $\epsilon = 0$. Integration by parts shows that

$$I_2 = -(n-1)\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} vH_i(v)F^{n-2}(v)F'(v)\,dv \ . \tag{8}$$

In order to calculate I_1 we first prove the following lemma:

Lemma 1

$$\sum_{i=1}^{n} \frac{dP_i(v)}{d\epsilon}\bigg|_{\epsilon=0} = (n-1)F^{n-2}(v)\sum_{i=1}^{n} H_i(v) \ .$$

Proof. Let $b_j(v)$ be the equilibrium strategy of player j. Differentiating the identity $v = b_j^{-1}(b_j(v;\epsilon);\epsilon)$ with respect to ϵ and substituting $\epsilon = 0$ gives

$$0 = \frac{\partial b_j^{-1}}{\partial \epsilon}\bigg|_{\epsilon=0} + (b_j^{-1})' \frac{\partial b_j}{\partial \epsilon}\bigg|_{\epsilon=0} \ . \tag{9}$$

Since in equilibrium

$$P_i(v) = P(b_i(v)) > \max_{j\neq i} b_j = \prod_{\substack{j=1 \\ j\neq i}}^{n} F_j(b_j^{-1}(b_i(v)) \ ,$$

we have that

$$\begin{aligned}
\frac{dP_i(v)}{d\epsilon}\bigg|_{\epsilon=0} &= F^{n-2}(v) \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial}{\partial \epsilon}\Big[F_j(b_j^{-1}(b_i(v))\Big]_{\epsilon=0} \\
&= F^{n-2}(v) \sum_{\substack{j=1 \\ j\neq i}}^{n} \left\{ \frac{\partial F_j}{\partial \epsilon}\bigg|_{\epsilon=0}(v) + F'(v)\frac{\partial}{\partial \epsilon}\Big[b_j^{-1}(b_i(v))\Big]_{\epsilon=0} \right\} \\
&= F^{n-2}(v) \sum_{\substack{j=1 \\ j\neq i}}^{n} \left\{ H_j(v) + F'(v)\left[\frac{\partial b_j^{-1}}{\partial \epsilon}\bigg|_{\epsilon=0} + (b^{-1})'\frac{\partial b_i(v)}{\partial \epsilon}\bigg|_{\epsilon=0}\right] \right\} \\
&= F^{n-2}(v) \sum_{\substack{j=1 \\ j\neq i}}^{n} \left\{ H_j(v) + F'(v)(b^{-1})'\left[-\frac{\partial b_j(v)}{\partial \epsilon}\bigg|_{\epsilon=0} + \frac{\partial b_i(v)}{\partial \epsilon}\bigg|_{\epsilon=0}\right] \right\} \ ,
\end{aligned}$$

where in the last stage we used (9). Summing this equation over $i = 1, \ldots, n$ and using the identities

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_i = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_j = (n-1)\sum_{i=1}^{n} a_i$$

completes the proof of the Lemma. ⊡

Using Lemma 1 we get that

$$\begin{aligned} I_1 &= -\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}} (n-1)F^{n-2}(v)H_i(v)[v(1-F(v))]'\,dv \\ &= -(n-1)\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}} F^{n-2}(v)H_i(v)[(1-F)-F'(v)v]\,dv \ . \end{aligned}$$

Combining this with (7),(8) gives (3). ⊡

3 Averaging

A powerful concept in perturbation theory is *averaging.* To illustrate this concept let us consider, for example, the calculation of heat transfer through a fluid-filled porous rock. Although the heat conductivity of the fluid and of the solid are known, an exact solution of the heat flow at each point in the rock is not possible. In most cases, however, one is only interested in the bulk properties of the rock and not in the microscopic variations of the heat flow. The idea behind averaging is to 'replace' the rock with a homogeneous medium whose bulk properties are 'similar'. Assuming that this is possible, the key question is how to correctly average the microscopic quantity of interest. For example, if the rock is 20% fluid and 80% solid, what is its bulk heat conductivity? It turns out that bulk properties are not always a simple arithmetic mean. For example, the bulk heat conductivity is the harmonic mean of the microscopic conductivities (see, e.g., Holmes (1995)).

In this section we apply similar averaging to asymmetric auctions, where the bulk property under consideration is the seller's expected revenue and the averaging is done over the asymmetry among players. To do that, let us first introduce the notations $R_{\text{sym}}[F]$ for the expected seller's revenue in *symmetric* auctions with n bidders with identical distribution function F, and $R[F_1, \ldots, F_n]$ for the expected seller's revenue in *asymmetric* auctions with distribution functions $\{F_i\}_{i=1}^n$. By the classical Revenue Equivalence Theorem

$$R_{\text{sym}}[F] = \bar{v} + (n-1)\int_{\underline{v}}^{\bar{v}} F^n dv - n\int_{\underline{v}}^{\bar{v}} F^{n-1}\, dv \ . \tag{10}$$

Our goal is to approximate the seller's expected revenue $R[F_1, \ldots, F_n]$ in asymmetric auctions as some average of expected revenue(s) in symmetric auctions, for which there exists the explicit expression (10). *A priori*, there are various possibilities to define the averaged revenue. For example:

1. We can use the arithmetic mean of the revenues, i.e., $R[F_1, \ldots, F_n] \approx \frac{1}{n}\sum_{i=1}^n R_{\text{sym}}[F_i]$.
2. We can use the geometric mean of the revenues, i.e., $R[F_1, \ldots, F_n] \approx \left(\prod_{i=1}^n R_{\text{sym}}[F_i]\right)^{1/n}$.
3. We can define the average distribution function as

   $$F_{\text{avg}} = \frac{1}{n}\sum_{i=1}^n F_i \ , \tag{11}$$

   and approximate

   $$R[F_1, \ldots, F_n] \approx R_{\text{sym}}[F_{\text{avg}}] \ . \tag{12}$$

4. We can define the average distribution function as the geometric mean $F_{\text{avg}} = \left(\prod_{i=1}^n F_i\right)^{1/n}$ and approximate $R[F_1, \ldots, F_n]$ using (12).

We now show that the correct averaging is given by (11),(12):

Theorem 2 *Let* $\epsilon = \max_i \max_v |F_i - F_{\rm avg}|$ *be small (i.e., weak asymmetry). Then the seller's expected revenue in any auction mechanism satisfying Conditions 1–4 is given by*

$$R[F_1, \ldots, F_n] = R_{\rm sym}[F_{\rm avg}] + O(\epsilon^2) \ ,$$

where $F_{\rm avg}$ *is given by (11).*

Proof. Apply Theorem 1 with $F = F_{\rm avg} = (1/n)\sum_{i=1}^n F_i$ and $H_i = (F_i - F_{\rm avg})/\epsilon$. Note that $R'(0) = 0$, since $\sum_{i=1}^n H_i(v) = 0$. $\boxdot$

Theorem 1 shows that in general the effect of asymmetry on the seller's expected revenue is of first order in ϵ. Thus, for example, when the functions F_i are given by (1), the naive approximation $R(\epsilon) \approx R(0) = R_{\rm sym}[F]$ has $O(\epsilon)$ accuracy. With the proper averaging (11),(12), however, the difference between $R(\epsilon)$ and $R_{\rm sym}[F_{\rm avg}]$ is only $O(\epsilon^2)$.

The result of Theorem 2 can also be interpreted as follows. Theorem 1 shows that the leading-order effect of asymmetry on the revenue R depends only on the *sum* of all the H_is. Hence, this leading-order effect would be the same if all players experience an identical (i.e., symmetric) change $F_i = F + \epsilon H$ (for $i = 1, \ldots, n$), provided that

$$H = \frac{1}{n}\sum_{i=1}^n H_i \ .$$

4 Second-price auctions

It is well known that in second-price auctions players have weak dominant strategies $b_i = v_i$ (Vickrey (1961)) such that the seller's expected revenue can be explicitly calculated by

$$R^{\text{2nd}} = E\left[\text{second }\max\{v_1, \ldots, v_n\}\right].$$

A simpler expression for R^{2nd} is derived in the following Lemma:

Lemma 2 *The expected revenue in a second-price auction is given by*

$$R^{\text{2nd}} = \overline{v} - \int_{\underline{v}}^{\overline{v}} \prod_{i=1}^{n} F_i(v)\, dv - \sum_{i=1}^{n} \int_{\underline{v}}^{\overline{v}} (1 - F_i(v)) \prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v)\, dv\ . \tag{13}$$

Proof. Let $\tilde{v}$ be the second highest number of the values $\{v_1, v_2, \ldots, v_n\}$. The distribution function of $\tilde{v}$ is given by

$$\begin{aligned} F^{\text{2nd}}(v) &= \Pr(\tilde{v} \leq v) = \Pr(v_1 \leq v, \ldots, v_n \leq v) + \sum_{i=1}^{n} \Pr(v_j \leq v, j \neq i, v_i > v) \\ &= \prod_{i=1}^{n} F_i(v) + \sum_{i=1}^{n} (1 - F_i(v)) \prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v). \end{aligned}$$

The expectation of $\tilde{v}$ is

$$\begin{aligned} R^{\text{2nd}} &= E(\tilde{v}) = \int_{\underline{v}}^{\overline{v}} v \frac{dF^{\text{2nd}}(v)}{dv}\, dv = vF^{\text{2nd}}(v)\big|_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} F^{\text{2nd}}(v)\, dv \\ &= \overline{v} - \int_{\underline{v}}^{\overline{v}} \left(\prod_{i=1}^{n} F_i(v) + \sum_{i=1}^{n} (1 - F_i(v)) \prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v) \right) dv\ , \end{aligned}$$

which gives (13). ⊡

If we substitute $F_i = F + \epsilon H_i$ in (13) we arrive at the result of Theorem 1, since

$$\begin{aligned} R^{2\mathrm{nd}} &= \bar{v} - \int_{\underline{v}}^{\overline{v}} \prod_{i=1}^{n} (F + \varepsilon H_i)\, dv - \sum_{i=1}^{n} \int_{\underline{v}}^{\overline{v}} (1 - F - \varepsilon H_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} (F + \varepsilon H_i)\, dv \\ &= \bar{v} - \int_{\underline{v}}^{\overline{v}} F^n\, dv - n \int_{\underline{v}}^{\overline{v}} (1 - F) F^{n-1}\, dv \\ &\qquad\qquad -\varepsilon(n-1) \int_{\underline{v}}^{\overline{v}} (1 - F) F^{n-2} \sum_{i=1}^{n} H_i\, dv + O(\varepsilon^2)\ . \end{aligned}$$

When there are only two players expression (13) reduces to

$$R^{2\mathrm{nd}} = \int_{\underline{v}}^{\overline{v}} (1 - F_1)(1 - F_2)\, dv\ .$$

In this case, substituting $F_i = F + \epsilon H_i$ gives the *exact* expression

$$R^{2\mathrm{nd}}(\epsilon) = \int_{\underline{v}}^{\overline{v}} (1 - F)^2\, dv - \epsilon \int_{\underline{v}}^{\overline{v}} (1 - F)(H_1 + H_2)\, dv + \epsilon^2 \int_{\underline{v}}^{\overline{v}} H_1 H_2\, dv\ . \tag{14}$$

5 Simulations

Unlike second-price auctions, explicit expressions of the equilibrium bids in first-price auctions exist only in the symmetric case. The equilibrium bids and the expected revenue, however, can be calculated numerically (see Marshall et al. (1994)). In this section we provide two examples that illustrate the results obtained in Theorems 1 and 2.

Example 1 *Consider the case of two bidders 1 and 2, whose valuations are distributed on* $[0, 1]$ *according to the distribution functions* $F_1 = v + \epsilon v(1 - v)$ *and* $F_2 = v - \epsilon v(1 - v)$*, respectively. Note that in this case* $F_{\mathrm{avg}} = v$ *and* $\sum_i H_i = 0$.

According to Theorem 1 we have that $R(0) = R_{\mathrm{sym}}[v] = 1/3$, $R'(0) = 0$, and thus that the seller's expected revenue is $R(\epsilon) = 1/3 + O(\epsilon^2)$. This result also follows from

ϵ	$\frac{R^{1st} - R_{\text{sym}}[F_{\text{avg}}]}{R_{\text{sym}}[F_{\text{avg}}]}$	$\frac{R^{2nd} - R_{\text{sym}}[F_{\text{avg}}]}{R_{\text{sym}}[F_{\text{avg}}]}$
0.05	-0.0036%	-0.025%
0.1	-0.015%	-0.1%
0.2	-0.054%	-0.4%
0.4	-0.172%	-1.6%

Table 1: Seller's expected revenue in asymmetric first-price R^{1st} and second-price R^{2nd} auctions as functions of ϵ, compared with the seller's expected revenue in the symmetric auction $R_{\text{sym}}[F_{\text{avg}}] = 1/3$.

Theorem 2, since in this case $F_{\text{avg}} = v$. The difference between the values of $R(\epsilon)$ and the approximation $R_{\text{sym}}[F_{\text{avg}}]$ can be seen in Table 1. Even for non-small values of ϵ, $R_{\text{sym}}[F_{\text{avg}}]$ is an excellent approximation of the expected revenue $R(\epsilon)$ for both first-price and second-price auctions. The simulation results also confirm that the error in the approximation of $R(\epsilon)$ with $R_{\text{sym}}[F_{\text{avg}}]$ scales like ϵ^2, as Theorem 2 predicts. Since the expected revenues in first-price and second-price auctions are not identical, the 'strict' revenue equivalence does not hold in this example. Indeed, whereas $R(0)$ and $R'(0)$ are identical among first-price and second-price auctions, the numerical results show that $R''(0)$ is different for first-price and second-price auctions, as $\frac{d^2}{d\epsilon^2} R^{2nd}\big|_{\epsilon=0} \approx -0.066 \neq \frac{d^2}{d\epsilon^2} R^{1st}\big|_{\epsilon=0} \approx -0.008$.

Example 2 *Consider the case of two bidders with distribution functions* $F_1 = v + \epsilon v(1-v)$ *and* $F_2 = v - 2\epsilon v(1-v)$*, respectively. Note that in this case* $\sum_i H_i \neq 0$ *and thus* $F_{\text{avg}} \neq v$ *except for the case when* $\epsilon = 0$*.*

ε	$R^{1st} - R_{\text{sym}}[F_{\text{avg}}]$	$R^{1st} - R_{\text{sym}}[v]$	$R^{2nd} - R_{\text{sym}}[F_{\text{avg}}]$	$R^{2nd} - R_{\text{sym}}[v]$
0.1	$-0.10 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$	$-0.75 \cdot 10^{-3}$	$7.67 \cdot 10^{-3}$
0.2	$-0.34 \cdot 10^{-3}$	$16.0 \cdot 10^{-3}$	$-3.0 \cdot 10^{-3}$	$14.0 \cdot 10^{-3}$
0.3	$-0.53 \cdot 10^{-3}$	$25.0 \cdot 10^{-3}$	$-6.75 \cdot 10^{-3}$	$19.0 \cdot 10^{-3}$
0.4	$-0.21 \cdot 10^{-3}$	$34.0 \cdot 10^{-3}$	$-12.0 \cdot 10^{-3}$	$22.0 \cdot 10^{-3}$

Table 2: Seller's expected revenue in asymmetric first-price R^{1st} and second-price R^{2nd} auctions as a function of ϵ compared with the seller's expected revenue in the symmetric auctions $R_{\text{sym}}[F_{\text{avg}}]$ and $R_{\text{sym}}[v]$.

In this example the bidders' distributions are not symmetric with respect to v, and the average distribution $F_{\text{avg}} = v - 0.5\epsilon v(1-v)$ depends on ϵ. From Theorem 1 we have that $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where $R(0) = R_{\text{sym}}[v] = \frac{1}{3}$ and $R'(0) = \frac{1}{12}$. Alternatively, from Theorem 2 we have that $R(\epsilon) = R_{\text{sym}}[v - 0.5\epsilon v(1-v)] + O(\epsilon^2)$ and by (10), $R_{\text{sym}}[v - 0.5\epsilon v(1-v)] = 1/3 + \epsilon/12 + O(\epsilon^2)$. Thus, the prediction for both R^{1st} and R^{2nd} is $R(\epsilon) = 1/3 + \epsilon/12$, with an $O(\epsilon^2)$ error.

In Table 2 we compare the expected revenue in the asymmetric case $R(\epsilon)$ with $R_{\text{sym}}[F_{\text{avg}}]$ and also with $R(0) = R_{\text{sym}}[v]$. As predicted, the two differences $R^{1st} - R_{\text{sym}}[F_{\text{avg}}]$ and $R^{2nd} - R_{\text{sym}}[F_{\text{avg}}]$ scale like ϵ^2, whereas $R^{1st} - R_{\text{sym}}[v]$ and $R^{2nd} - R_{\text{sym}}[v]$ scale only like ϵ.

In conclusion, the above examples demonstrate that when asymmetry is weak, the difference in revenue between a first-price auction and a second-price auction is negligible. Therefore asymmetric first-price auctions and second-price auctions remain essentially revenue equivalent.

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