Dominant Strategies, Superior Information, and Winner's Curse in Second-Price Auctions[*,†]

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Abstract

In a general model of common-value second-price auctions with differential information, we show equivalence between the following characteristics of a bidder: (i) having a dominant strategy; (ii) possessing superior information; (iii) being immune from winner's curse. When a dominant strategy exists, it is given by the conditional expectation of the common value with respect to bidder's information field; if the dominant strategy is used, other bidders cannot make a profit.

Keywords*:* common-value second-price auctions, differential information, dominant strategies, information superiority, winner's curse.

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1 Introduction

In a second-price sealed-bid auction, each bidder independently submits a bid. The bidder with the highest bid wins the auctioned object and pays the second-highest bid. The other bidders receive nothing but do not incur any costs.[1] It is well known that in private-value second-price auctions, bidding his value of the object is a dominant strategy for every bidder (e.g., Vickrey (1961)). In common-value second-price auctions with asymmetric information (first studied by Wilson (1977)) the situation is more complicated, and dominant strategies may not exist.

This paper focuses on a characterization of the circumstances under which dominant strategies exist in common-value second price auctions. Specifically, we consider the pure form of common-value auctions with differential information, where the value is identical for all bidders in the same state of nature, but the information about which state of nature was realized can differ. Our main result, Theorem 3, establishes the equivalence between the possession of a dominant strategy by a bidder and the superiority of this bidder's information. More precisely, we show that a bidder has a dominant strategy in an auction if and only if his information allows him to predict the value of the auctioned object as capably as if he had the aggregate information of all bidders.[2]

If a bidder has a dominant strategy, it is uniquely determined and consists of bidding the expected value of the object conditional on this bidder's information (see Proposition 1). This, coupled with our equivalence theorem, provides a superiorly informed bidder (who can be, for instance, the auction's insider) with a straightforward and simple bidding strategy. And, in the contexts where existence of superiorly informed bidders is unlikely (e.g., when bidders' estimates are imperfect, and independent conditional on the true value), the equivalence theorem is in effect an "impossibility" result, ruling out the existence of dominant strategies.

Despite the fact that bidder's dominant strategy is given by his *unbiased* estimate of the true value, this strategy is never prone to winner's

[1] See, for example, Vickrey (1961), McAfee and McMillan (1987), Wilson (1992) and Klemperer (1999).

[2] Our notion of information superiority also manifests itself in terms of resulting outcomes. We show in Theorem 5 that, when a bidder with superior information uses his dominant strategy, other bidders cannot make a profit: their conditional expected payoffs are non-positive in all states of nature.

curse (to avoid which, bidders are typically advised to bias their estimates downwards[3]). More precisely, the dominant strategy is immune from three common variants of the curse:[4] the bidder's expected profit is always non-negative, even after is revealed to him that his bid is the highest; the bidder's expected value of the object does not decrease following this revelation; and, finally, the revelation also does not diminish the bidder's expected profit. It turns out, moreover, that the immunity from winner's curse (of either kind) precisely demarcates the cases in which the bidder has a dominant strategy. We show in Theorems 9, 10, and 11 that the unbiased estimate strategy is immune from winner's curse if and only if it is dominant (or, equivalently, if and only if the bidder possesses superior information).

Finally, let us note that our model of differential information is very general, and that it imposes no restrictions on the space of states of nature or on the information fields of bidders. In particular, our framework allows for continuous as well as discrete private information. The results are therefore valid in the classical set-up[5], where bidders receive informational signals, as well as in the set-up where information endowments are represented by partitions in the space of states of nature.

2 The Model

2.1 Common-Value Second-Price Auctions

Consider a group of *bidders* $N = \{1, 2, ..., n\}$ who compete to acquire an indivisible object through a second-price auction. The value of the object is the same for all the bidders but there is uncertainty about the common value of the object. This uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is the set of *states of nature*, $\mathcal{F}$ is a σ-field of subsets of Ω, and μ is a σ-additive probability measure on $(\Omega, \mathcal{F})$. Once the state of nature ω is realized, the common value $v(\omega)$ of the object is determined. To reduce the number of technicalities, it is assumed that the *value function* $v : \Omega \to R_+$ is bounded from above.

Let $x = (x_1, ..., x_n) \in R_+^n$ be a vector of bids submitted by members of N,

[3] See, e.g., McAffee and McMillan (1987), p. 271.

[4] We discuss these variants in Section 5, based on survey papers of McAffee and McMillan (1987) and Thaler (1988).

[5] I.e., the model of Milgrom and Weber (1982a) with the common value assumption.

and denote by $m(x) = |\{i \in N \mid x_i = \max_{j \in N} x_j\}|$ the number of bidders tied at the highest bid. The payoff (or profit) of bidder i is given by the function $u_i : \Omega \times R^n_+ \to R$:

$$u_i(\omega, x) = \begin{cases} \frac{1}{m(x)}(v(\omega) - \max_{j \in N \setminus \{i\}} x_j) & \text{if } \; x_i = \max_{j \in N} x_j; \\ 0 & \text{otherwise} \end{cases} .$$

While the bidders do not observe the state of nature that actually occurs, they may have some information about it. The initial information of bidder i is described by a σ- subfield $\mathcal{F}_i$ of $\mathcal{F}$, his *information field.* That is, given any $\mathcal{F}_i$-measurable set, bidder i knows whether the realized state of nature is included in this set.[6]

A common-value second-price auction with differential information is thus described by the collection $(N, (\Omega, \mathcal{F}, \mu), v, \{u_i\}_{i=1}^n, \{\mathcal{F}_i\}_{i=1}^n)$.

2.2 Strategies, Dominance Relation, and Equilibrium

A (pure) *strategy* for bidder i in the auction is a bounded function $b_i : \Omega \to R_+$ which is also $\mathcal{F}_i$-measurable. We denote by S_i the set of all strategies of bidder i, and by $S = \times_{j \in N} S_j$ the set of all strategy profiles. Also put $S_{-i} = \times_{j \in N \setminus \{i\}} S_j$.

If X is an integrable random variable on Ω, and H is a σ-subfield of $\mathcal{F}$, we denote by $E(X \mid H)$ the conditional expectation of X with respect to H (see, e.g., Section 34 of Billingsley (1995)). If H is a finite field, that is, a field consisting of unions of disjoint sets $\{M_1, ..., M_m\} \in \mathcal{F}$ such that $\cup_{j=1}^m M_j = \Omega$, then $E(X \mid H)$ is given simply by

$$E(X \mid H)(\omega) = E(X \mid M_j),$$

where M_j is the set that contains ω, and $E(X \mid M_j)$ is the expectation of X conditional on the event M_j. Our boundedness assumptions on the value function and bidders' strategies guarantees that for all $i \in N$ and all $b \in S$, i's expected payoff conditional on H, $E(u_i(\cdot, b(\cdot)) \mid H)$, is defined μ-almost everywhere.

[6] If, as in the model of Milgrom and Weber (1982a), bidder i's infromation is represented by the knowledge of an $\mathcal{F}$-measurable variable X_i, then $\mathcal{F}_i$ is simply the minimal σ-subfield of $\mathcal{F}$ with respect to which X_i is measurable.

We say that strategy b_i of bidder i *dominates* his strategy c_i if for all $b_{-i} \in S_{-i}$ and almost all $\omega \in \Omega$,

$$E(u_i(\cdot,(b_i(\cdot),b_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) \geq E(u_i(\cdot,(c_i(\cdot),b_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) \quad (1)$$

and there exist $\widehat{b}_{-i} \in S_{-i}$ and an event A with $\mu(A) > 0$ such that for all $\omega \in A$,

$$E(u_i(\cdot,(b_i(\cdot),\widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) > E(u_i(\cdot,(c_i(\cdot),\widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega). \quad (2)$$

Strategy b_i of bidder i is *dominant* if it dominates any other strategy of bidder i which differs from b_i on a positive probability event.

A (pure strategy) *Bayesian equilibrium* of the auction is a profile of strategies $\widehat{b} = (\widehat{b}_1, ..., \widehat{b}_n) \in S$ such that for every $i \in N$, $b_i \in S_i$, and almost every $\omega \in \Omega$

$$E(u_i(\cdot,\widehat{b}(\cdot)) \mid \mathcal{F}_i)(\omega) \geq E(u_i(\cdot,(b_i(\cdot),\widehat{b}_{-i}(\cdot)) \mid \mathcal{F}_i)(\omega).$$

Note that if b, c are strategy profiles in S and $b_i = c_i$ almost everywhere, then $u_i(\cdot, b(\cdot)) = u_i(\cdot, c(\cdot))$ almost everywhere. Thus we can (and will) identify between strategies which are equal almost everywhere. Further, by $f = g$, $f > g$, or $f \geq g$ we will mean that $f(\omega) = g(\omega)$ or $f(\omega) \geq g(\omega)$ for almost every $\omega \in \Omega$; and saying that $f = g$, $f > g$, or $f \geq g$ on event A will mean that the equality or inequality hold pointwise for almost every $\omega \in A$.

3 Dominant Strategies and Information Superiority

We start by noting that dominant strategies of bidders, provided that they exist, have a very specific form: a bidder bids the expected value of the object, conditional on information that was revealed to him (Proposition 1 below). To prove this, we generalize well known arguments which show, in the complete information case, that bidder's dominant strategy is to bid the actual value.

Proposition 1 *If $i \in N$ has a dominant strategy, it is equal to $E(v \mid \mathcal{F}_i)$.*

Proof. Denote $b_i = E\left(v \mid \mathcal{F}_i\right)$. Let c_i be a dominant strategy of i, and suppose contrary to the assertion that $b_i \neq c_i$. We will show that there is $\widehat{b}_{-i} \in S_{-i}$ and an event A with $\mu(A) > 0$ such that on A

$$E(u_i(\cdot, (b_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i) > E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i). \tag{3}$$

It follows from $b_i \neq c_i$ and $\mathcal{F}_i$-measurability of b_i and c_i that there is $C \in \mathcal{F}_i$ of positive measure such that either $b_i > c_i$ on C, or $b_i < c_i$ on C. Suppose first that $b_i > c_i$ on C. Since the union of $\mathcal{F}_i$-measurable sets of the form $C_r = \{\omega \in C \mid b_i(\omega) > r > c_i(\omega)\}$, for all rational r, is equal to C, it follows that there exists r_0 for which $\mu\left(C_{r_0}\right) > 0$. Denote $A = C_{r_0}$.

Now take $\widehat{b}_j\left(\omega\right) \equiv r_0$ for every $j \in N \setminus \{i\}$, $\omega \in \Omega$. For all $\omega \in A$

$$u_i(\omega, (b_i(\omega), \widehat{b}_{-i}(\omega))) = v(\omega) - r_0$$

and

$$u_i(\omega, (c_i(\omega), \widehat{b}_{-i}(\omega))) = 0.$$

Thus, on A

$$E(u_i(\cdot, (b_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i) = E(v \mid \mathcal{F}_i) - r_0 > 0 =$$

$$= E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i),$$

which establishes (3). But this contradicts our assumption that c_i is a dominant strategy of i.

Now, if $c_i > b_i$ on C, we can construct r_0, A, and $\widehat{b}_{-i}$ by the method used above. Thus, on A

$$E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i) = E(v \mid \mathcal{F}_i) - r_0 < 0 =$$

$$= E(u_i(\cdot, (b_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i).$$

This implies (3), in contradiction to the assumed dominance of c_i. ■

The concept of bidder's information superiority, that we now introduce, extends the natural definition, according to which the bidder's information field must include information fields of all other bidders. The need to extend this definition is explained in the following example.

Example 1 Let $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $v(\omega_i) = i$ for $i = 1, 2$ and $v(\omega_3) = 2$. Let $\mathcal{F}_1$ be the minimal field on Ω containing $\{\omega_1\}, \{\omega_2, \omega_3\}$, and $\mathcal{F}_2$ be the minimal field on Ω containing $\{\omega_1, \omega_2\}, \{\omega_3\}$. Clearly $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$. However, bidder 1 always knows the real value of the object, unlike bidder 2. It would be difficult not to interpret this fact as an expression of 1's information superiority.

This brings us to:

Definition 2 *Denote by* $\bigvee_{j \in N} \mathcal{F}_j$ *the minimal* σ*-field containing all* $\mathcal{F}_j$*. We say that* $i \in N$ *has superior information in the auction if*

$$E\left(v \mid \bigvee_{j \in N} \mathcal{F}_j\right) = E\left(v \mid \mathcal{F}_i\right) \tag{4}$$

The interpretation of (4) is simple: the equation says that the information of bidder i allows him to predict the value of the auctioned object just as well as if he had the aggregate information of all the bidders. Having the largest information field implies (4), and hence it is a particular form of information superiority.

Theorem 3 below shows the equivalence between possession of a dominant strategy and possession of superior information by a bidder.

Theorem 3 *Bidder* $i \in N$ *has a dominant strategy if and only if he has superior information.*

Proof. I. We start by treating the "if" direction. We will show first that for every $c_i \in S_i$ and $b_{-i} \in S_{-i}$,

$$\max\left\{E(v \mid \bigvee_{j \in N} \mathcal{F}_j) - d, 0\right\} \geq E(u_i(\cdot, (c_i(\cdot), b_{-i}(\cdot))) \mid \bigvee_{j \in N} \mathcal{F}_j), \tag{5}$$

where $d = \max_{j \in N \setminus \{i\}} b_j$. Indeed, since $m(c_i, b_{-i})$, d, and $\mathbf{1}_{d \leq c_i}$ (the indicator function of the event $\{\omega \in \Omega \mid d(\omega) \leq c_i(\omega)\}$) are $\bigvee_{j \in N} \mathcal{F}_j$-measurable,

$$E(u_i(\cdot, (c_i(\cdot), b_{-i}(\cdot)) \mid \bigvee_{j \in N} \mathcal{F}_j) = E(\frac{1}{m\left(c_i, b_{-i}\right)} \left[v - d\right] \cdot \mathbf{1}_{d \leq c_i} \mid \bigvee_{j \in N} \mathcal{F}_j) = \tag{6}$$

$$= \frac{1}{m(c_i, b_{-i})} \left[E(v \mid \bigvee_{j \in N} \mathcal{F}_j) - d \right] \cdot \mathbf{1}_{d \leq c_i} \leq \max \left\{ E(v \mid \bigvee_{j \in N} \mathcal{F}_j) - d, 0 \right\}. \tag{7}$$

This implies (5).

Now, given $c_i \in S_i$, $c_i \neq E(v \mid \mathcal{F}_i)$, we have to show that $b_i = E(v \mid \mathcal{F}_i)$ dominates c_i. Let $b_{-i} \in S_{-i}$. Looking at the proof of (5) with b_i instead of c_i, we see that the inequality in (7) holds as equality, since

$$E(v \mid \bigvee_{j \in N} \mathcal{F}_j) - d = E(v \mid \mathcal{F}_i) - d = b_i - d,$$

and if $b_i - d > 0$ then $m(b_i, b_{-i}) = 1$. Thus

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot)) \mid \bigvee_{j \in N} \mathcal{F}_j) = \max \left\{ E(v \mid \bigvee_{j \in N} \mathcal{F}_j) - d, 0 \right\}, \tag{8}$$

which, together with (5), implies

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot)) \mid \bigvee_{j \in N} \mathcal{F}_j) \geq E(u_i(\cdot, (c_i(\cdot), b_{-i}(\cdot)) \mid \bigvee_{j \in N} \mathcal{F}_j).$$

Since for every $\mathcal{F}$-measurable and integrable random variable X

$$E\left(E\left(X \mid \bigvee_{j \in N} \mathcal{F}_j\right) \mid \mathcal{F}_i\right) = E\left(X \mid \mathcal{F}_i\right) \tag{9}$$

(e.g., Theorem 34.4 in Billingsley (1995)), the above inequality leads to

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i) \geq E(u_i(\cdot, (c_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i),$$

which is (1) in the definition of dominance. In order to complete the proof that b_i is a dominant strategy for bidder i, we have to show (2). This, however, can be established precisely as (3) in the proof of Proposition 1.

II. Now we prove the "only if" direction of the theorem. Assume, contrary to the assertion, that bidder i has a dominant strategy (which must be equal to $E(v \mid \mathcal{F}_i)$ by Proposition 1) and yet (4) does not hold. We claim that for all $\varepsilon, \delta > 0$ small enough there are sets $S^j \in \mathcal{F}_j$ for every $j \in N$ and a positive number a such that:

(i) $|E(v \mid \mathcal{F}_i)(\omega) - a| < \varepsilon$ for every $\omega \in S^i$;

(ii) $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)(\omega) < a - 10\varepsilon$ for ω in a certain $\bigvee_{j\in N} \mathcal{F}_j$-measurable subset B of $S^1 \cap ... \cap S^n$ with $\mu(B) \geq (1-\delta)\,\mu(S^1 \cap ... \cap S^n)$;

(iii) $\mu(S^1 \cap ... \cap S^n) > 0$.

First note that, if for some ε and every set of the form $\{\omega \in \Omega \mid |E(v \mid \mathcal{F}_i)(\omega) - a| < \varepsilon\}$, the subset of ω for which $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)(\omega) < a - 10\varepsilon$ is of measure zero, then

$$E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) - E(v \mid \mathcal{F}_i) \geq -11\varepsilon$$

(almost everywhere). If this is the case for all ε belonging to some vanishing sequence, we have $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \geq E(v \mid \mathcal{F}_i)$. However, the equality

$$E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)\right) = E(v) = E(E(v \mid \mathcal{F}_i))$$

yields $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) = E(v \mid \mathcal{F}_i)$, which contradicts our assumption that (4) does not hold.

This shows that for all sufficiently small ε there is a set $S' = \{\omega \mid |E(v \mid \mathcal{F}_i)(\omega) - a| < \varepsilon\}$ such that $T' = \left\{\omega \in S' \mid E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)(\omega) < a - 10\varepsilon\right\}$ is of positive measure. Since T' is $\bigvee_{j\in N} \mathcal{F}_j$-measurable, it can be approximated in measure by sets generating the field $\bigvee_{j\in N} \mathcal{F}_j$, i.e., unions (without loss of generality, disjoint) of intersections of N sets, from each one of the $\mathcal{F}_j$'s. Thus, for any $\delta > 0$ there are $S_1^j, ..., S_m^j \in \mathcal{F}_j$ (suppose also that $S_k^i \subset S'$) such that

$$\mu\left(\left[T' \backslash \bigcup_{k=1}^{m} S_k^1 \cap ... \cap S_k^n\right] \cup \left[\bigcup_{k=1}^{m} S_k^1 \cap ... \cap S_k^n \backslash T\right]\right) < \delta\mu\left(\bigcup_{k=1}^{m} S_k^1 \cap ... \cap S_k^n\right),$$

and $\mu(S_k^1 \cap ... \cap S_k^n) > 0$ for every k. Thus there exists k such that

$$\mu\left(\left(S_k^1 \cap ... \cap S_k^n\right) \backslash T'\right) < \delta\mu\left(S_k^1 \cap ... \cap S_k^n\right).$$

Define $S^j = S_k^j$ and $B = S_k^1 \cap ... \cap S_k^n \cap T'$; it is easy to see that these sets satisfy conditions (i)-(iii) above.

Now consider strategies b_j of $j \in N \backslash \{i\}$ given by

$$b_j(\omega) = \begin{cases} a - 3\varepsilon, & \text{if } \omega \in S^j; \\ a + 10\varepsilon, & \text{otherwise} \end{cases} . \tag{10}$$

Recall that the dominant strategy $b_i = E(v \mid \mathcal{F}_i)$ of i takes values in the interval $(a - \varepsilon, a + \varepsilon)$ on the set S^i. It follows that, given S^i, the dominant strategy of i only succeeds in winning the auction on $S^1 \cap ... \cap S^n$ against b_{-i}. Thus on B

$$E\left(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \bigvee_{j \in N} \mathcal{F}_j\right) \leq (a - 10\varepsilon) - (a - 3\varepsilon) = -7\varepsilon. \tag{11}$$

The definition of the conditional expectation and (11) yield

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid B) =$$

$$= E\left(E\left(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \bigvee_{j \in N} \mathcal{F}_j\right) \mid B\right) \leq -7\varepsilon.$$

For small enough δ this implies that

$$E\left(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid S^1 \cap ... \cap S^n\right) < 0 \tag{12}$$

(we use boundedness of u_i). Since $u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) = 0$ on $S^i \backslash (S^1 \cap ... \cap S^n)$ (clearly bidder i loses the auction at these states), also

$$E\left(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid S^i\right) < 0.$$

However,

$$E\left(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid S^i\right) = E\left(E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i) \mid S^i\right),$$

and so $E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i)$ must be negative on a subset A of S^i with $\mu(A) > 0$.

Now consider a strategy c_i of bidder i, which is equal to $a - 4\varepsilon$ on S^i. Using this strategy against b_{-i} means a loss in the auction in all states of S^i. Thus $E(u_i(\cdot, c_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i) = 0$ on S^i, and so

$$E(u_i(\cdot, (c_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i) > E(u_i(\cdot, b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i)$$

on A. This contradicts (1) in the definition of b_i as a dominant strategy. ■

We already noted that (4) is satisfied when bidder i's information field is the largest. The following is therefore a corollary of Theorem 3.

Corollary 4 *If $i \in N$ is such that $\mathcal{F}_j \subset \mathcal{F}_i$ for every $j \in N$, then i has a dominant strategy.*

4 Information Superiority and Payoffs

In the context of first-price auctions, Milgrom (1979) and Milgrom and Weber (1982b, Theorem 3) showed that, in presence of a bidder with superior information, expected payoffs of all other bidders are zero in any Bayesian equilibrium. Despite that this is not always the case in second price auctions (see Example 2 below), our next result shows that information disadvantage of bidders is reflected in outcomes of auctions where a bidder with superior information uses his dominant strategy. Precisely, we have:

Theorem 5 *If $i \in N$ has superior information then for every $b_{-i} \in S_{-i}$ and $j \in N \backslash \{i\}$*

$$E(u_j(\cdot,(b_i^*(\cdot),b_{-i}(\cdot)) \mid \mathcal{F}_j)(\omega) \leq 0,$$

where b_i^* denotes i's dominant strategy (that is, $b_i^* = E\left(v \mid \mathcal{F}_i\right)$).

Proof. Let $j \in N \backslash \{i\}$, and denote $d = \max_{k \in N \backslash \{i,j\}}(b_i^*, b_k)$. Functions $m\left(b_i^*, b_{-i}\right)$, d and $\mathbf{1}_{b_j \geq d}$ are $\bigvee_{k \in N} \mathcal{F}_k$-measurable, and so

$$E(u_j(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid \bigvee_{k \in N} \mathcal{F}_k) = E(\frac{1}{m\left(b_i^*, b_{-i}\right)}\left[v-d\right] \cdot \mathbf{1}_{b_j \geq d} \mid \bigvee_{k \in N} \mathcal{F}_k) =$$

$$= \frac{1}{m\left(b_i^*, b_{-i}\right)}\left[E(v \mid \bigvee_{k \in N} \mathcal{F}_k) - d\right] \cdot \mathbf{1}_{b_j \geq d}.$$

By (4) in the definition of superior information, this expression is equal to $\frac{1}{m(b_i^*, b_{-i})}\left[b_i^* - d\right] \cdot \mathbf{1}_{b_j \geq d}$, which is non-positive. Thus

$$E(u_j(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid \bigvee_{k \in N} \mathcal{F}_k) \leq 0,$$

and, after applying (9),

$$E(u_j(\cdot,(b_i^*(\cdot),b_{-i}(\cdot)))\mid \mathcal{F}_j)=E(E(u_j(\cdot,(b_i^*(\cdot),b_{-i}(\cdot)))\mid \bigvee_{k\in N}\mathcal{F}_k)\mid \mathcal{F}_j)\leq 0.$$

■

The intuition behind Theorem 5 is simple. In order to beat bidder i with superior information, other bidders have to bid more. Bids in excess of i's bid, however, are never below the object's expected value given bidders' aggregate information ($\equiv i$'s bid), and thus cannot lead to positive expected payoffs.

The following example shows that, in a Bayesian equilibrium of common-value second-price auction with differential information, the expected payoff of a bidder with inferior information can be positive, and, in fact, it can exceed the payoff of a bidder with superior information. In light of Theorem 5, however, this phenomenon is possible only because the superiorly informed bidder does not use his dominant strategy in the equilibrium.

Example 2 Let $N=\{1,2\}$, $\Omega=\{\omega_1,\omega_2\}, \mathcal{F}=2^{\Omega}, \mu(\{\omega_j\})=\frac{1}{2}$ and $v(\omega_j)=j$ for $j=1,2$. The information field of bidder 1 is $\mathcal{F}_1=\mathcal{F}$ and the information field of bidder 2 is $\mathcal{F}_2=\{\emptyset,\Omega\}$ (i.e., bidder 2 is completely uninformed).

Consider a pair strategies $b=(b_1,b_2)\in S$, where $b_1(\omega)=1$ and $b_2(\omega)=2$ for all $\omega\in\Omega$. It is easy to verify that b is a Bayesian equilibrium of G. It is clear that, under b, 1 loses the auction and 2 wins it at every $\omega\in\Omega$, paying 1. We obtain that for all $\omega\in\Omega$,

$$E(u_1(\cdot,b(\cdot))\mid \mathcal{F}_1)(\omega)=0,$$

and

$$E(u_2(\cdot,b(\cdot))\mid \mathcal{F}_2)(\omega)=\frac{1}{2}((1-1)+(2-1))=\frac{1}{2}.$$

That is,

$$E(u_2(\cdot,b(\cdot))\mid \mathcal{F}_2)>E(u_1(\cdot,b(\cdot))\mid \mathcal{F}_1)=0.$$

5 Dominant Strategy and Winner's Curse

The term *winner's curse* refers to a possible disappointment that the bidder who wins the auction can experience. The reference is often to the situation in which the winning bidder receives negative profits, meaning that he would have been better off if he had lost the auction in the first place. (Literature on the topic is abound; see, e.g., surveys of Thaler (1988) and Laffont (1997)). However, other sources of dissatisfaction are clearly imaginable. Bidder may get disappointed because the value of the object that he won was less than the value he expected before he was pronounced a winner (see, e.g., Thaler (1988) and McAfee and McMillan (1987)). Or, the value of the object net of the payment to the auctioneer (i.e., the actual profit) can decrease compared to the profit expected by the bidder before his victory.

It is commonly suggested that a bidder should bid less then his unbiased estimate of the object's value ($E(v \mid \mathcal{F}_i)$), in order to protect himself from winner's curse (see, e.g., p. 721 of McAffy and McMillan (1987)). But under what circumstances the strategy $b_i^* = E(v \mid \mathcal{F}_i)$ is nevertheless immune from the winner's curse? We will show that the immunity of this strategy is tantamount to its dominance (or, information superiority of i). Moreover, this equivalence prevails under any of the three mentioned variants of the concept of winner's curse, despite that the concepts are not equivalent in general.

To lay ground for formal definitions, we start from several notations. Given $b \in S$ and $i \in N$, let $B_i = \left\{\omega \in \Omega \mid b_i(\omega) \geq \max_{j \in N \setminus \{i\}} b_j(\omega)\right\}$ (the event that i wins the auction). Also, denote by $\mathcal{F}_i \vee B_i$ the σ-subfield of $\mathcal{F}$ generated by $\mathcal{F}_i$ and the set B_i. Thus, $\mathcal{F}_i \vee B_i$ is the information field that allows bidder i to distinguish whether he won or lost the auction, in addition to his original information endowment (represented by $\mathcal{F}_i$).

Definition 6 *Strategy b_i of $i \in N$ is prone to the first variant of winner's curse (W1), given $b_{-i} \in S_{-i}$, if*

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) < 0 \tag{13}$$

on some $\mathcal{F}_i \vee B_i$-measurable $A \subset B_i$ with $\mu(A) > 0$. We say that b_i is immune from W1 if it is not prone to W1 given any b_{-i}, that is

$$\forall b_{-i} \in S_{-i}: \;\; E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) \geq 0 \tag{14}$$

on B_i.

Our second definition of winner's curse describes it in terms of diminishment in bidder's expected value of the object, when it is revealed to him that his bid is the highest.

Definition 7 *Strategy b_i of $i \in N$ is prone to the second variant of winner's curse (W2), given $b_{-i} \in S_{-i}$, if*

$$E(v \mid \mathcal{F}_i \vee B_i) < E(v \mid \mathcal{F}_i) \tag{15}$$

on some $\mathcal{F}_i \vee B_i$-measurable $A \subset B_i$ with $\mu(A) > 0$. We say that b_i is immune from W2 if it is not prone to W2 given any b_{-i}, that is

$$\forall b_{-i} \in S_{-i}: \;\; E(v \mid \mathcal{F}_i) \leq E(v \mid \mathcal{F}_i \vee B_i) \tag{16}$$

on B_i.

According to our third definition of winner's curse, bidder's expected payoff, rather than value, diminish when it is revealed to him that his bid is the highest.

Definition 8 *Strategy b_i of $i \in N$ is prone to the third variant of winner's curse (W3), given $b_{-i} \in S_{-i}$, if*

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) < E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i) \tag{17}$$

on some $\mathcal{F}_i \vee B_i$-measurable $A \subset B_i$ with $\mu(A) > 0$. We say that b_i is immune from W3 if it is not prone to W3 given any b_{-i}, that is

$$\forall b_{-i} \in S_{-i}: \;\; E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i) \leq E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) \tag{18}$$

on B_i.

We start from the following theorem, which shows the promised equivalence between immunity from W1 and the dominance of b_i^* (or, superiority of i's information).

Theorem 9 *Strategy b_i^* is dominant for bidder $i \in N$ if and only if it is immune from W1. Equivalently, bidder i has superior information if and only if b_i^* is immune from W1.*

Proof. We will only prove the second assertion in the theorem; the first assertion will follow from Theorem 3 and Proposition 1.

Assume first that bidder i has superior information. Reasoning as in part I of the proof of Theorem 3 we can show that

$$E(u_i(\cdot,(b_i^*(\cdot),b_{-i}(\cdot)) \mid \bigvee_{j\in N} \mathcal{F}_j) = \max\left\{E(v \mid \bigvee_{j\in N} \mathcal{F}_j) - \max_{j\in N\setminus\{i\}} b_j, 0\right\}$$

for every $b_{-i} \in S_{-i}$ (this is equality (8) in that proof). This implies (14), and so b_i^* is indeed immune from W1.

Next assume b_i^*'s immunity from W1, and suppose that, contrary to the assertion, bidder i does not have superior information.

Given sufficiently small $\varepsilon, \delta > 0$, we consider $a > 0$ and sets $S^j \in \mathcal{F}_j$ ($j \in N$) which satisfy conditions (i)-(iii), listed in part II of the proof of Theorem 3. Existence of such sets was established in that proof (based on the assumption that i does not have superior information). Further, let $b_{-i} \in S_{-i}$ be given by (10). Thus, $S^i \cap B_i = S^1 \cap ... \cap S^n$, $\mu(S^i \cap B_i) > 0$, and

$$E(u_i(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid S^1 \cap ... \cap S^n) < 0$$

if δ is small enough (this inequality is precisely (12), shown in the proof of Theorem 3). Since

$$E(u_i(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid S^1 \cap ... \cap S^n) = E(u_i(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid S^i \cap B_i)$$

$$= E\left(E(u_i(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) \mid S^i \cap B_i\right),$$

it follows that

$$E(u_i(\cdot,(b_i^*(\cdot),b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) < 0 \qquad (19)$$

on some A, an $\mathcal{F}_i \vee B_i$-measurable subset of $S^i \cap B_i$ with $\mu(A) > 0$. However, (19) contradicts (14), which shows that i indeed has superior information. ■

Now, we prove the equivalence with immunity from W2.

Theorem 10 *Theorem 9 holds with W2 instead of W1.*

Proof. Assume first that bidder i has superior information. In this case

$$E\left(v \mid \mathcal{F}_i \vee B_i\right) = E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \mid \mathcal{F}_i \vee B_i\right)$$

$$= E\left(E\left(v \mid \mathcal{F}_i\right) \mid \mathcal{F}_i \vee B_i\right) = E\left(v \mid \mathcal{F}_i\right).$$

Thus (16) clearly holds, and so b_i^* is immune form W2.

Now, assume that b_i^* is immune from W2. Note that, for any positive measure set of the form $S^1 \cap ... \cap S^n$, where $S^j \in \mathcal{F}_j$ for every $j \in N$, there are $b_j \in S_j$ such that $S^i \cap B_i = S^1 \cap ... \cap S^n$ (e.g., let $b_j = 0$ on S^j, and $b_j = \sup_\Omega v + 1$ otherwise). Since

$$E\left(v \mid S^1 \cap ... \cap S^n\right) = E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \mid S^1 \cap ... \cap S^n\right),$$

$$E\left(v \mid S^1 \cap ... \cap S^n\right) = E\left(E\left(v \mid \mathcal{F}_i \vee B_i\right) \mid S^1 \cap ... \cap S^n\right),$$

and

$$E\left(v \mid \mathcal{F}_i \vee B_i\right) \geq E\left(v \mid \mathcal{F}_i\right)$$

on B_i, we see that

$$E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \mid S^1 \cap ... \cap S^n\right) \geq E\left(E\left(v \mid \mathcal{F}_i\right) \mid S^1 \cap ... \cap S^n\right).$$

Since finite unions of disjoint sets of the form $S^1 \cap ... \cap S^n$ are an algebra that generates the σ-field $\bigvee_{j\in N} \mathcal{F}_j$, we have, in fact, that

$$E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \mid A\right) \geq E\left(E\left(v \mid \mathcal{F}_i\right) \mid A\right)$$

for any $A \in \bigvee_{j\in N} \mathcal{F}_j$ with $\mu(A) > 0$. It follows that

$$E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \geq E(v \mid \mathcal{F}_i),$$

which, together with the equality

$$E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \mid \mathcal{F}_i\right) = E(v \mid \mathcal{F}_i),$$

implies (4). Thus i indeed has superior information. ■

Finally, we show that immunity from W3 also characterizes existence of a dominant strategy, or information superiority:

Theorem 11 *Theorem 9 holds with W3 instead of W1.*

Proof. Before we begin, note that if X is a bounded and $\mathcal{F}$-measurable random variable which vanishes on the complement of B_i, then

$$E(X \mid \mathcal{F}_i) = \mu(B_i \mid \mathcal{F}_i)E(X \mid \mathcal{F}_i \vee B_i) \tag{20}$$

on B_i (where $\mu(B_i \mid \mathcal{F}_i)$ stands for the probability of B_i conditional on $\mathcal{F}_i$). This claim is easily verified for X which are indicator functions of sets $S \cap B_i$, where $S \in \mathcal{F}_i$, and for X which are linear combinations of these indicator functions. Since a general $\mathcal{F}_i \vee B_i$-measurable X can be approximated almost everywhere by a bounded sequence of the mentioned linear combinations, the validity of (20) for such X follows from Theorem 34.2 (v) of Billingsley (1995). Finally, if X is merely $\mathcal{F}$-measurable, X can be replaced by an $\mathcal{F}_i \vee B_i$-measurable function $E(X \mid \mathcal{F}_i \vee B_i)$ without changing the equality (Theorem 34.4 of Billingsley (1995)).

Assume now that i has superior information. By Theorem 9, b_i^* is immune from W1. Then (18) follows from (20), by taking $X = u_i(\cdot, (b_i^*(\cdot), b_{-i}(\cdot)))$, and so b_i^* is immune from W3 as well.

Assume next that b_i^* is immune from W3, and suppose, contrary to the assertion, that i does not have superior information. As in second part of the proof of Theorem 9, we can construct $b_{-i} \in S_{-i}$ and an $\mathcal{F}_i \vee B_i$-measurable set $A \subset B_i$ with $\mu(A) > 0$ such that (19) will hold on it. Also, by the definition of b_{-i} (which goes back to (10) in the proof of Theorem 3) and the construction of A, $m(b_i^*, b_{-i}) = 1$ on A.

By taking $X = u_i(\cdot, (b_i^*(\cdot), b_{-i}(\cdot)))$ in (20), (19) implies (17) which is a contradiction to the immunity from W3, unless $\mu(B_i \mid \mathcal{F}_i) = 1$ on some $\mathcal{F}_i \vee B_i$-measurable subset $A' \subset A$ with positive measure.

If $\mu(B_i \mid \mathcal{F}_i) = 1$ on A', however, we also have

$$E\left(v \mid \mathcal{F}_i\right) = E\left(v \mid \mathcal{F}_i \vee B_i\right)$$

on A'. Thus, on A',

$$E(u_i(\cdot, (b_i^*(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i \vee B_i) \geq E\left(v \mid \mathcal{F}_i \vee B_i\right) - b_i^*$$

$$= E\left(v \mid \mathcal{F}_i \vee B_i\right) - E\left(v \mid \mathcal{F}_i\right) = 0,$$

which contradicts (19). This contradiction proves that b_i^* is also immune from W1. ■

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