

Contests with Reservation Prices*

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Abstract

We study contests where n privately informed players bid for a single prize. All players bear a cost of bidding that is an increasing function of their bids. We show that setting a reservation price is always profitable for a seller who wishes to maximize the highest bid. On the other hand, it might not be profitable to set a reservation price for a seller who wishes to maximize the average bid. However, if players have exogenous entry costs (i.e., the cost of preparing a bid) then, independent of the values of the entry costs, setting a reservation price is always profitable.

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1 Introduction

Several studies in the literature illustrate that it is profitable to limit the number of players in contests. For example, Nalebuff and Stiglitz (1983) showed that in labor tournaments, permitting too many workers to compete for a prize may reduce the overall level of effort. Taylor (1995) argued that in research tournaments with homogeneous players, excessive players reduce the total level of research effort. In a model of a research tournament, Fullerton and McAfee (1999) proved that the optimal number of players should be two. Reservation prices, or, alternatively, entry fees, are often used in contests to limit the entry of players. By setting a reservation price, the seller deters the entry of players whose expected profit is less than the level of the reservation price, such that he can raise the level of the players' efforts.

In this paper we focus on the question of whether in contests that are isomorphic to all-pay auctions it is profitable for the seller to exclude players with low valuations for the item being sold by imposing a reservation price. We consider n players who compete for a single item. Every player submits and pays a bid for the item being sold, but only the highest bidder receives the item. All players bear a cost of bidding that is an increasing function of their bids. Each player's valuation for the item is drawn independently from the same interval according to a distribution function that is the same for all the players and that is common knowledge.¹ Each player is privately informed about his valuation. The seller can

¹Note that the question addressed here of whether it is profitable for the seller to impose a reservation price is relevant only if the seller's valuation and the bidders' valuations are drawn approximately from the same range.

impose a reservation price such that a player who wants to participate in the contest has to place a bid that is larger or equal to this reservation price.²

Let us refer to the situation of a contest without a reservation price as the status quo. The effect of a reservation price on the equilibrium bids in the status quo is ambiguous. On the one hand, a reservation price has a negative effect on the equilibrium bids of players with relatively low valuations that are excluded from the contest, such that their bids become lower (their bids are actually equal to zero) than the bids in the status quo. On the other hand, a reservation price has a positive effect on the equilibrium bids of players with relatively high valuations who participate in the contest such that their bids become higher than the bids in the status quo. Indeed we show that in all-pay auctions, where the seller wishes to maximize the highest bid, it is always profitable for the seller to set a reservation price. However, in all-pay auctions where the seller wishes to maximize the average bid it is not clear whether setting a reservation price is profitable for the seller. If the negative effect of the reservation price on the equilibrium bids of players with low valuations will be higher than its positive effect on the equilibrium bids of players with high valuations, setting a reservation price is not profitable for the seller. When the situation is reversed, it is profitable to set a reservation price.

According to the seminal studies of Myerson (1981) and Riley and Samuelson (1981), if bidders have linear cost functions, the classical private-value auctions with the optimal

²More information about our model of all-pay auctions with incomplete information can be found in Moldovanu and Sela (2001).

reservation price are revenue-maximizing among all feasible mechanisms.³ In particular, this result implies that in our model, if players have linear cost functions, setting a reservation price is profitable for the seller who wishes to maximize the average bid. However, we show that this result cannot be generalized for the case where players do not have linear cost functions, and a seller who wishes to maximize the average bid may not necessarily set a reservation price.⁴ We provide a sufficient condition under which setting a reservation price is profitable for a seller in first-price all-pay auctions.

Finally, we assume that players have exogenous entry costs that may be interpreted as the cost of preparing a bid. That is, participating in the contest generates a fixed cost for each player from which the seller does not benefit. In this case, we show that in all-pay auctions, regardless of the value of the entry costs, setting a reservation price is profitable for a seller who wishes to maximize either the highest bid or the average bid. Hence, our results demonstrate that in practice a reservation price or any alternative instrument that can exclude players with low valuations should be used in every all-pay contest.

³Feasible mechanisms are mechanisms that are incentive-compatible and individually rational.

⁴In such cases where the cost functions are non-linear and setting a reservation price is not profitable for the seller, Gaviols, Moldovanu and Sela (2001) found that capping the bids can be alternative for setting a reservation price and it may be profitable for the seller. (for more information on bid caps, see also Che and Gale (1998)).

2 The model

Consider n players bidding for an indivisible item. Player i 's valuation for the item, v_i , is drawn independently from the interval $[0, 1]$ according to the distribution function F , and it is private information to i , $i = 1, 2, \dots, n$.⁵ We assume that F is continuously differentiable and it is common knowledge. The player with the highest bid wins the item, but all the n players pay their bids. A bid x causes a cost $c(x)$, where $c : R_+ \rightarrow R_+$ is a strictly increasing function, twice continuously differentiable with $c(0) = 0$. If more than one player submits the highest bid, then the winner is randomly selected among the highest bidders, i.e., each bidder has the same chance to win the item.

The seller's valuation for the item v_0 is drawn from the interval $[0, 1]$.⁶ The seller can impose a reservation price $E > 0$ such that a player who wants to participate in the contest has to place a bid that is larger or equal to E . In the following we divide the contests for two classes (A and B) according to the seller's aims; in class A the seller wishes to maximize the highest bid and in class B the seller wishes to maximize the average bid.

3 Equilibrium

The symmetric equilibrium bid function (see, for example, Klemperer (1999)) in an all-pay auction where players have a linear cost function is given by

$$b(v) = vF^{n-1}(v) - \int_0^v F^{n-1}(y)dy, \quad 0 \leq v \leq 1. \quad (1)$$

⁵The choice of the interval $[0, 1]$ is a normalization.

⁶All our results hold also if v_0 is not much bigger than 1 or is not much smaller than 0.

The equilibrium bid functions where players have a non-linear cost function $c(\cdot)$ is $c^{-1}(b(v))$, where $b(v)$ is given by (1).

Assume now that the seller imposes a reservation price E . In a symmetric equilibrium, the bid function of every player is given by $c^{-1}(b_E(v))$, where

$$b_E(v) = vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy, \quad 0 \leq v \leq 1 \quad (2)$$

and $h \in [0, 1]$ is the critical value which is the lowest type who participates in the contest.

The critical value h is determined by the condition

$$c^{-1}(b_E(h)) = c^{-1}(hF^{n-1}(h)) = E.$$

4 Class A - the highest bid

In every contest in class A the seller wishes to maximize the highest bid. In that case, the seller's expected payoff is given by

$$V_{high} = \int_h^1 c^{-1} \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) nF^{n-1}(v)f(v)dv + v_0F^n(h).$$

The following result shows that in every contest in class A , independent of the number of players, the form of the cost functions and the seller's valuation for the item being sold, setting a reservation price is always effective.

Proposition 1 *In every all-pay auction, setting a reservation price is profitable for the seller who wishes to maximize the highest bid.*

Proof: See Appendix.

While the result given by Proposition 1 seems intuitive, it is not clear whether it is effective to set a reservation price in all-pay auctions where the seller maximizes the average bid. This is discussed in the next section.

5 Class B - the average bid

In every contest in Class B the seller wishes to maximize the average bid. In that case, the seller's expected payoff is given by

$$V_{aver} = n \int_h^1 c^{-1}(b_E(v)) f(v) dv + v_0 F^n(h).$$

In all-pay auctions where players have linear cost functions setting a reservation price is profitable for the seller (Myerson (1981)). In the following we show that this result cannot be generalized to the case when players do not have linear cost functions.

Proposition 2 *In all-pay auctions, setting a reservation price might not be profitable for the seller who wishes to maximize the average bid.*

Proof:

Consider an all-pay auction where $F(x) = x^\alpha$, $\alpha > 0$ and $c(x) = x^\beta$, $\beta > 1$. Below we show that if n is large enough, setting a reservation price is not profitable for a seller with a valuation $0 \leq v_0 \leq 1$.

The first order condition of the seller's expected profit is given by

$$\begin{aligned}
\frac{1}{n} \frac{\partial V_{aver}}{\partial h} &= \\
&= F^{n-1}(h) \int_h^1 (c^{-1})' \left(v F^{n-1}(v) - \int_h^v F^{n-1}(y) dy \right) f(v) dv \\
&\quad - c^{-1}(h F^{n-1}(h)) f(h) + v_0 F^{n-1}(h) f(h) \\
&= h^{(n-1)\alpha} \int_h^1 \frac{1}{\beta} \left(\frac{(n-1)\alpha v^{(n-1)\alpha+1}}{(n-1)\alpha+1} + \frac{h^{(n-1)\alpha+1}}{(n-1)\alpha+1} \right)^{\frac{1-\beta}{\beta}} \alpha v^{\alpha-1} dv \\
&\quad - (h^{(n-1)\alpha+1})^{\frac{1}{\beta}} \alpha h^{\alpha-1} + \alpha v_0 h^{(n-1)\alpha} h^{\alpha-1} \\
&< \frac{\alpha}{\beta} \left(\frac{(n-1)\alpha}{(n-1)\alpha+1} \right)^{\frac{1-\beta}{\beta}} h^{(n-1)\alpha} \int_h^1 v^{((n-1)\alpha+1)\frac{1-\beta}{\beta} + \alpha - 1} dv \\
&\quad - \alpha h^{((n-1)\alpha+1)\frac{1}{\beta} + \alpha - 1} + \alpha v_0 h^{n\alpha-1} \\
&= \frac{\alpha}{\beta} \left(\frac{(n-1)\alpha}{(n-1)\alpha+1} \right)^{\frac{1-\beta}{\beta}} \frac{h^{(n-1)\alpha}}{((n-1)\alpha+1)\frac{1-\beta}{\beta} + \alpha} \left(1 - h^{((n-1)\alpha+1)\frac{1-\beta}{\beta} + \alpha} \right) \\
&\quad - \alpha h^{((n-1)\alpha+1)\frac{1}{\beta} + \alpha - 1} + \alpha v_0 h^{n\alpha-1} \\
&= \alpha K(n) \left(h^{((n-1)\alpha+1)\frac{1}{\beta} + \alpha - 1} - h^{(n-1)\alpha} \right) - \alpha h^{((n-1)\alpha+1)\frac{1}{\beta} + \alpha - 1} + \alpha v_0 h^{n\alpha-1} \\
&= \alpha(K(n) - 1) h^{((n-1)\alpha+1)\frac{1}{\beta} + \alpha - 1} - \alpha K(n) h^{(n-1)\alpha} + \alpha v_0 h^{n\alpha-1}
\end{aligned}$$

where

$$K(n) = \frac{1}{\beta} \left(\frac{(n-1)\alpha}{(n-1)\alpha+1} \right)^{\frac{1-\beta}{\beta}} \frac{1}{((n-1)\alpha+1)\frac{\beta-1}{\beta} - \alpha}. \quad (3)$$

Observe that $K(n) > 0$ and $K(n) \sim \frac{1}{n}$. Thus, for a large enough n

$$K(n) < 1. \quad (4)$$

Moreover, since $\frac{1}{\beta} < 1$, for a large enough n , $h^{n\alpha-1} \ll h^{((n-1)\alpha+1)\frac{1}{\beta} + \alpha - 1}$ and then we have

$\frac{\partial V}{\partial h} < 0$ for every h . *Q.E.D*

In the proof of Proposition 2 it is assumed that the number of players who participate in the auction is sufficiently large. The following example shows that setting a reservation

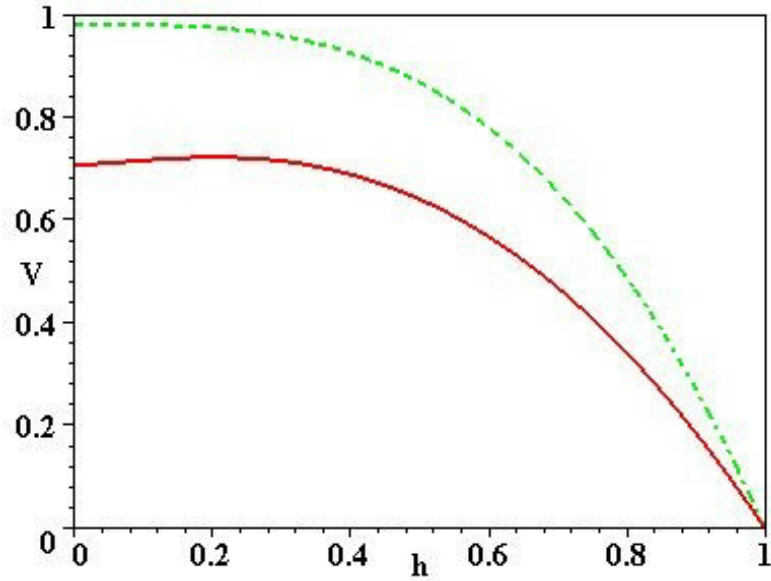


Figure 1:

price may not be profitable for the seller even if only a few players participate in the auction.

Example 1 Consider an all-pay auction where $v_0 = 0$, $F(x) = x$ and $g(x) = x^2$.

We will show that if the number of players is larger than $n = 3$, it is not profitable for the seller to set a reservation price.

The seller's expected revenue is given by

$$V_{aver} = n \int_h^1 \left(\frac{n-1}{n} v^n + \frac{1}{n} h^n \right)^{\frac{1}{2}} dv.$$

As Figure 1 shows, the seller's expected revenue in the above auction with three players is larger than his expected payoff in the same auction with two players. When $n = 2$, the optimal h is around 0.2, however, when $n = 3$ the optimal h is zero.

By (3) and (4), $\frac{\partial V_{aver}}{\partial h} < 0$ for every h if

$$\sqrt{\frac{n}{n-1} \frac{1}{n-2}} < 1.$$

Note that for $n > 3$, $\frac{\partial V_{aver}}{\partial h} < 0$ for every h , and therefore it is profitable for the seller not to set a reservation price. Moreover, a numerical calculation shows that even for $n = 3$, it is profitable for the seller not to set a reservation price.

Proposition 2 shows that in all-pay auctions exclusion of players with low valuations may not be effective. However, the following result provides the conditions under which a reservation price increases the seller's expected profit.

Proposition 3 *If $c'(0) > 0$, setting a reservation price is always profitable for the seller who wishes to maximize the average bid.*

Proof: See Appendix.

According to Proposition 3 we conclude that

Corollary 1 *If $c(\cdot)$ is concave, setting a reservation price is profitable for the seller.*

Proof: If $c(\cdot)$ is concave, then $c'(0) > 0$ and the result follows directly from Proposition 3. *Q.E.D*

According to Proposition 3, a reservation price may not be effective when $c'(0) = 0$ or alternatively $(c^{-1})'(0) = \infty$. This condition implies that the equilibrium bids of players with low valuations increases extremely such that the equilibrium bids of players with low valuations are not negligible with respect to the equilibrium bids of the rest of players and

therefore it is not profitable for the seller to exclude any player. In the next section we show that, independent of the form of the cost function, a reservation price may be profitable for a seller who wishes to maximize the average bid in all-pay auctions.

6 Exogenous entry costs

Consider an all-pay auction in which the seller sets a reservation price E . Assume that each player has an external entry cost K which may be interpreted as the cost of preparing a bid. In other words, participating in the auction generates a fixed cost K for each player independently of any reservation price that the seller may dictate. In this case, the seller does not benefit from the extra cost K . K and E obey the following relation

$$K + E = c^{-1}(b_E(v)) = c^{-1}(hF^{n-1}(h))$$

where h is the critical value, and the equilibrium bids $c^{-1}(b_E(v))$ are the same as in the previous section.

The previous section indicates that setting a reservation price might not be profitable for the seller who wishes to maximize the average bid. The following result shows that in the case of exogenous entry costs, a seller should impose a reservation price in every all-pay auction regardless of the value of the players' exogenous costs.

Proposition 4 *If players have an exogenous entry cost $K > 0$, setting a reservation price is always profitable for the seller.*

Proof: See Appendix.

It is of interest to note that Baye et al. (1993) showed that in all-pay auctions with complete information and heterogenous agents, excluding specific participants can be beneficial. According to Proposition 4, we prove that if players have external costs, excluding participants in all-pay auctions with incomplete information is always beneficial.

7 Appendix

7.1 Proof of Proposition 1

In order to find the optimal h , we differentiate V_{high} with respect to h :

$$\begin{aligned} \frac{\partial V_{high}}{\partial h} &= nF^{n-1}(h) \int_h^1 (c^{-1})' \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) F^{n-1}(v)f(v)dv \\ &\quad - nc^{-1}(hF^{n-1}(h))F^{n-1}(h)f(h) + nv_0F^{n-1}(h)f(h) \\ &= nF^{n-1}(h) \left[\int_h^1 (c^{-1})' \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) F^{n-1}(v)f(v)dv - c^{-1}(hF^{n-1}(h))f(h) + \right. \end{aligned}$$

Since c^{-1} is an increasing function and $c^{-1}(0) = 0$, we obtain

$$\left[\int_h^1 (c^{-1})' \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) F^{n-1}(v)f(v)dv - c^{-1}(hF^{n-1}(h))f(h) + v_0f(h) \right]_{h=0} > 0.$$

Thus, for every h near zero we find that $\frac{\partial V}{\partial h} > 0$. This implies that setting a reservation price is profitable for the seller. *Q.E.D*

7.2 Proof of Proposition 3

The expected revenue in an all-pay auction where the seller wishes to maximize the average bid is given by

$$V_{aver} = n \int_h^1 c^{-1} \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) f(v)dv + v_0F^n(h).$$

The first order condition of the seller's expected profit is given by

$$\begin{aligned} \frac{1}{n} \frac{\partial V_{aver}}{\partial h} &= F^{n-1}(h) \int_h^1 (c^{-1})' \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) f(v)dv \\ &\quad - c^{-1}(hF^{n-1}(h))f(h) + v_0F^{n-1}(h)f(h) \\ &= 0 \end{aligned} \quad (5)$$

We proceed to show that if $c'(0) > 0$, then $\frac{\partial V_{aver}}{\partial h} > 0$ for every h that is close enough to zero. Dividing (5) by $hF^{n-1}(h)$ yields

$$\begin{aligned} &\frac{1}{h} \left[\int_h^1 (c^{-1})' \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) f(v)dv + v_0f(h) \right] \\ &= \frac{c^{-1}(hF^{n-1}(h))}{hF^{n-1}(h)} f(h) = (c^{-1})'(x)f(h) \end{aligned} \quad (6)$$

where $x \in (0, hF^{n-1}(h))$. Note that since $c'(0) > 0$, when h approaches zero, the right side of (6) is bounded while the left side approaches infinity. Thus, $\frac{\partial V_{aver}}{\partial h} > 0$ near zero, which implies that there is a reservation price which is profitable for the seller. *Q.E.D*

7.3 Proof of Proposition 4

Denote h_0 as the solution to

$$K = c^{-1} (h_0F^{n-1}(h_0)). \quad (7)$$

As before, setting a reservation price E is the same as setting the critical value h , where h is constrained by $h_0 \leq h$. Note that since each player's bid includes the entry cost K , the seller's profit from the participants at the auction is the total expected sum of bids minus the exogenous entry costs as follows:

$$n \int_h^1 \left(c^{-1} \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) - K \right) f(v)dv \quad (8)$$

From (7) and (8) we find that the seller's expected revenue is

$$V_{aver} = n \int_h^1 \left(c^{-1} \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) - c^{-1} (h_0F^{n-1}(h_0)) \right) f(v)dv + v_0F^n(h). \quad (9)$$

Dividing (9) by n and differentiating with respect to h (recall that $h_0 \leq h$) yields

$$\begin{aligned} \frac{1}{n} \frac{\partial V_{aver}}{\partial h} &= F^{n-1}(h) \int_h^1 (c^{-1})' \left(vF^{n-1}(v) - \int_h^v F^{n-1}(y)dy \right) f(v)dv + \\ &c^{-1} (h_0F^{n-1}(h_0)) f(h) - c^{-1} (hF^{n-1}(h)) f(h) + v_0F^{n-1}(h)f(h). \end{aligned}$$

When $h = h_0$ we have

$$\frac{1}{n} \frac{\partial V_{aver}}{\partial h} \Big|_{h=h_0} = F^{n-1}(h_0) \left[\int_{h_0}^1 (c^{-1})' \left(vF^{n-1}(v) - \int_{h_0}^v F^{n-1}(y)dy \right) f(v)dv + v_0f(h_0) \right] > 0.$$

Thus, setting a reservation price $E > 0$ is profitable for the seller. *Q.E.D*

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